STEP/MAT/AEA Questions by A Level Chapter (Pure)

The following questions are aligned to the chapters of Pearson's A Level textbooks for the new 2017 A Level Maths. I've only included a question in a chapter if students would have already covered *all* the skills involved, hence why chapters towards the end of the textbook have more associated questions in this compilation.

CONTENTS

1 YEAR 1

1.1 ALGEBRAIC EXPRESSIONS

Question 1 (STEP I 2006 Q1)

Find the integer, n, that satisfies $n^2 < 33127 < (n+1)^2$. Find also a small integer m such that $(n+m)^2 - 33127$ is a perfect square. Hence express 33127 in the form pq, where p and q are integers greater than 1.

By considering the possible factorisations of 33127, show that there are exactly two values of m for which $(n+m)^2 - 33127$ is a perfect square, and find the other value.

Solutions: (i) 157×211 (ii) $m = 16382$

Question 2 (STEP I 2006 Q6)

(i) Show that, if $(a\,,b)$ is **any** point on the curve $x^2 - 2y^2 = 1$, then $(3a+4b\,,2a+3b)$ also lies on the curve.

(ii) Determine the smallest positive integers M and N such that, if $(a\,,b)$ is **any** point on the curve $Mx^2 - Ny^2 = 1$, then $(5a+6b\,,4a+5b)$ also lies on the curve.

(iii) Given that the point $(a\,,b)$ lies on the curve $x^2 - 3y^2 = 1$, find positive integers P, Q, R and S such that the point $(Pa+Qb\,,Ra+Sb)$ also lies on the curve.

Question 3 (STEP I 2005 Q7i,ii)

The notation $\displaystyle\prod_{r=1}^{n} \mathrm{f}(r)$ denotes the product $\mathrm{f}(1) \times \mathrm{f}(2) \times \mathrm{f}(3) \times \cdots \times \mathrm{f}(n)$.

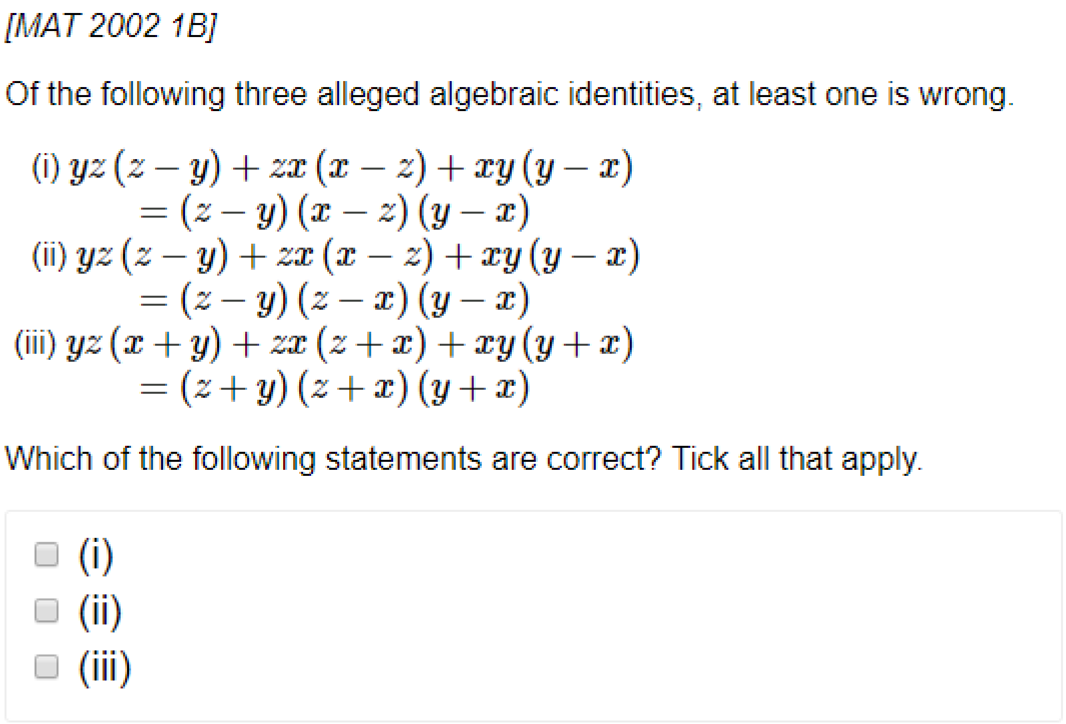
Simplify the following products as far as possible:

(i) $\displaystyle\prod_{r=1}^{n} \left(\frac{r+1}{r}\right)$;

(ii) $\displaystyle\prod_{r=2}^{n} \left(\frac{r^2-1}{r^2}\right)$;

Solutions: (i) $n+1$ (ii) $\frac{n+1}{2n}$

Question 4



[MAT 2002 1B]

Of the following three alleged algebraic identities, at least one is wrong.

(i) $yz(z-y)+zx(x-z)+xy(y-x)$
$= (z-y)(x-z)(y-x)$

(ii) $yz(z-y)+zx(x-z)+xy(y-x)$
$= (z-y)(z-x)(y-x)$

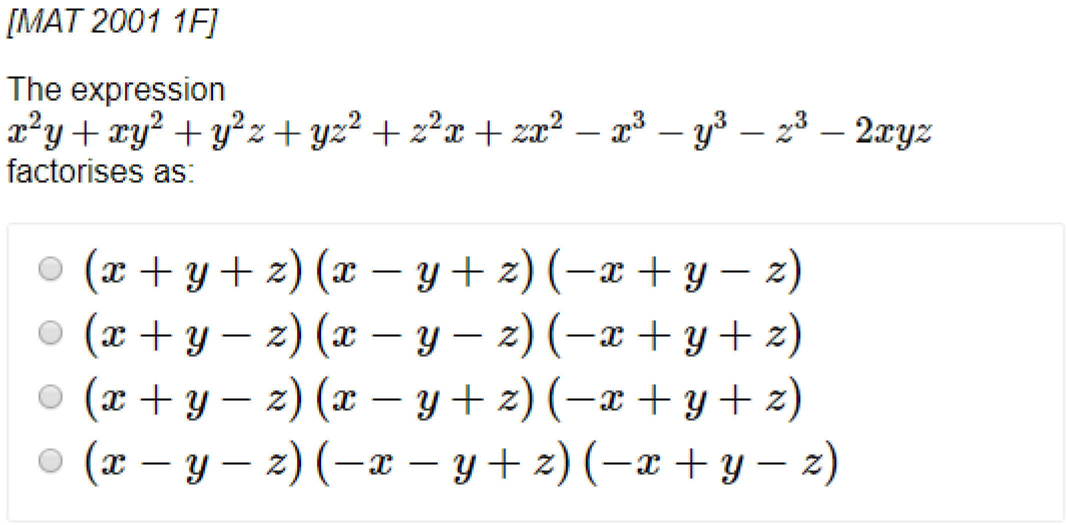
(iii) $yz(x+y)+zx(z+x)+xy(y+x)$
$= (z+y)(z+x)(y+x)$

Which of the following statements are correct? Tick all that apply.

- ☐ (i)
- ☐ (ii)
- ☐ (iii)

Solution: (ii) only

Question 5



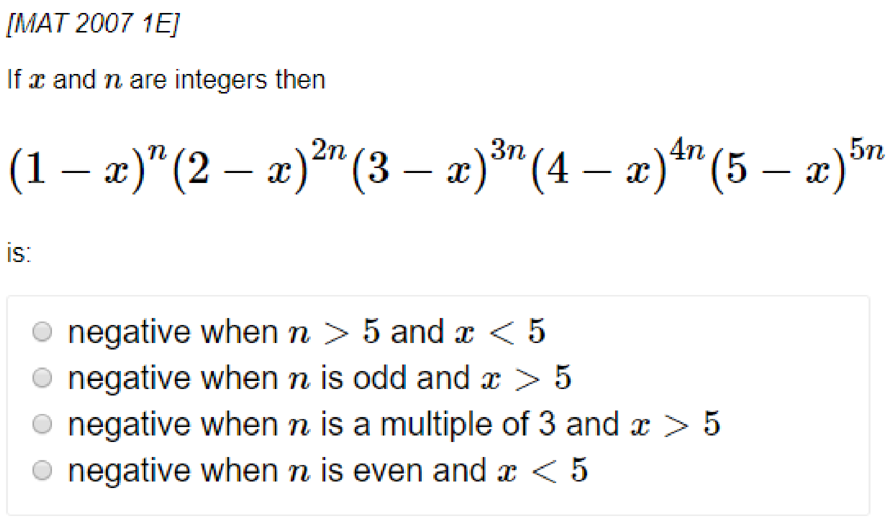
[MAT 2001 1F]

The expression
$x^2y+xy^2+y^2z+yz^2+z^2x+zx^2-x^3-y^3-z^3-2xyz$
factorises as:

- ○ $(x+y+z)(x-y+z)(-x+y-z)$
- ○ $(x+y-z)(x-y-z)(-x+y+z)$
- ○ $(x+y-z)(x-y+z)(-x+y+z)$
- ○ $(x-y-z)(-x-y+z)(-x+y-z)$

Solution: Option 3.

Question 6



[MAT 2007 1E]

If x and n are integers then

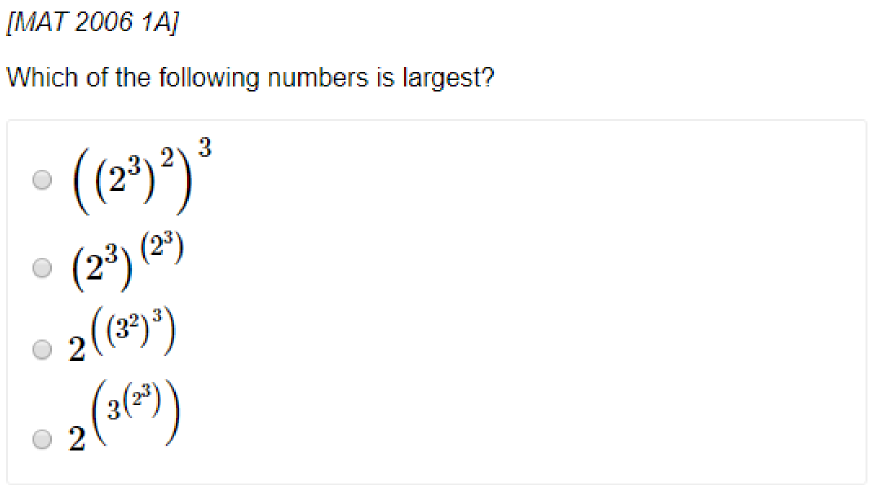
$$(1-x)^n(2-x)^{2n}(3-x)^{3n}(4-x)^{4n}(5-x)^{5n}$$

is:

- ○ negative when $n > 5$ and $x < 5$
- ○ negative when n is odd and $x > 5$
- ○ negative when n is a multiple of 3 and $x > 5$
- ○ negative when n is even and $x < 5$

Solution: Option 2.

Question 7



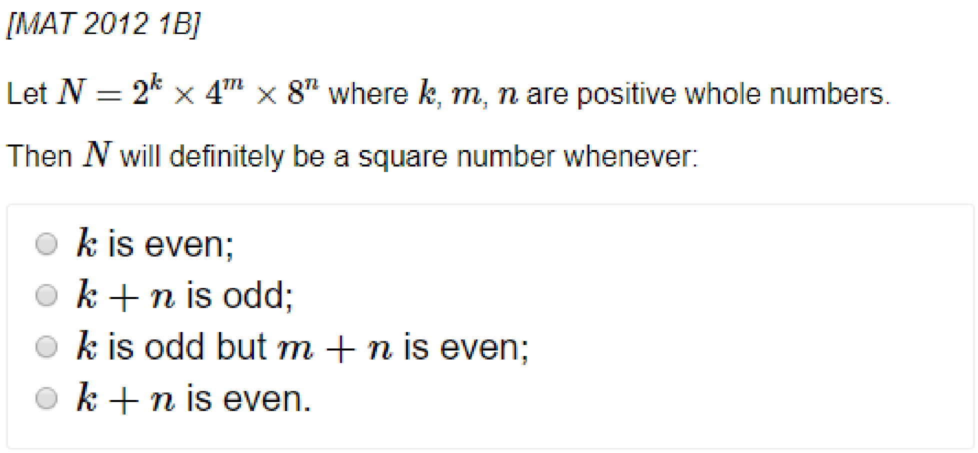
[MAT 2006 1A]

Which of the following numbers is largest?

- ○ $\left((2^3)^2\right)^3$
- ○ $(2^3)^{(2^3)}$
- ○ $2^{\left((3^2)^3\right)}$
- ○ $2^{\left(3^{(2^3)}\right)}$

Solution: Option 4.

Question 8



[MAT 2012 1B]

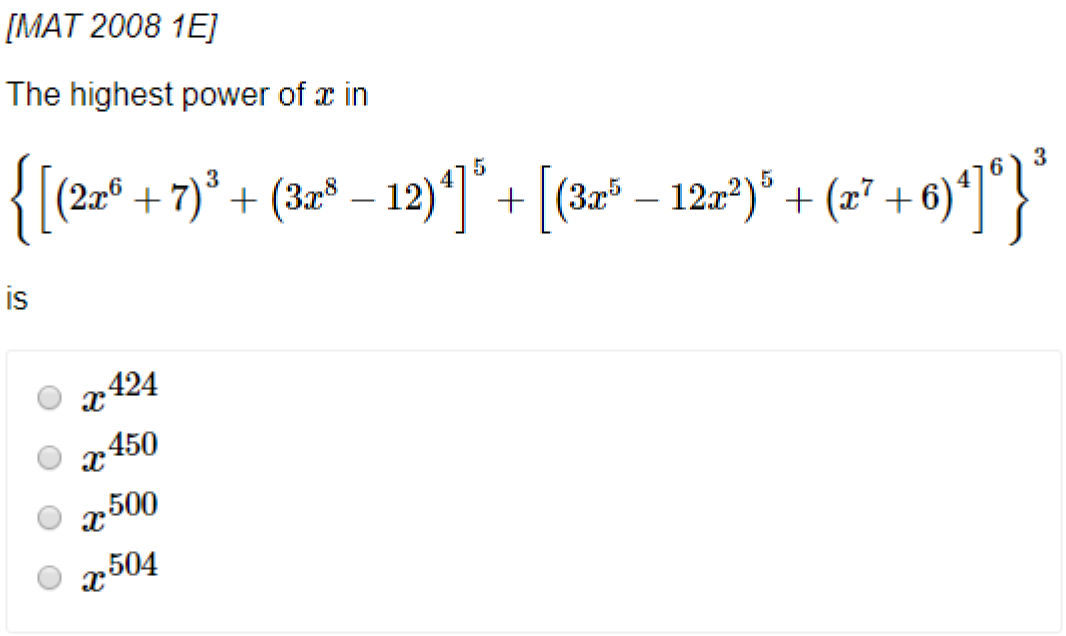
Let $N = 2^k \times 4^m \times 8^n$ where k, m, n are positive whole numbers.

Then N will definitely be a square number whenever:

- ○ k is even;
- ○ $k + n$ is odd;
- ○ k is odd but $m + n$ is even;
- ○ $k + n$ is even.

Solution: Option 4.

Question 9



[MAT 2008 1E]

The highest power of x in

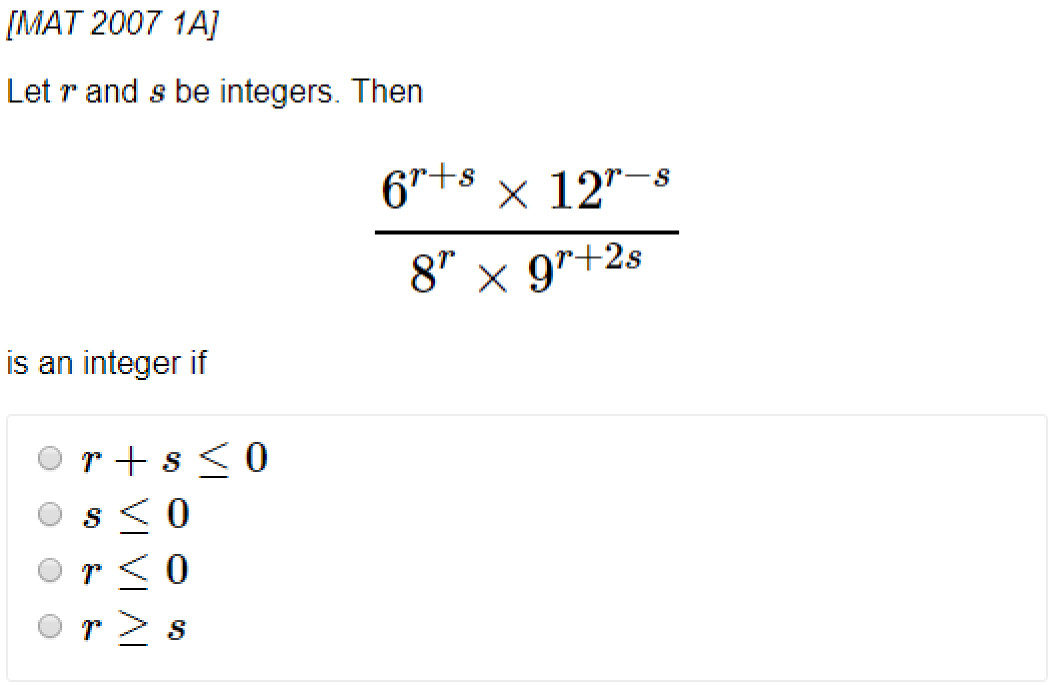
$$\left\{\left[\left(2x^6+7\right)^3+\left(3x^8-12\right)^4\right]^5+\left[\left(3x^5-12x^2\right)^5+\left(x^7+6\right)^4\right]^6\right\}^3$$

is

○ x^{424}
○ x^{450}
○ x^{500}
○ x^{504}

Solution: x^{504}

Question 10



[MAT 2007 1A]

Let r and s be integers. Then

$$\frac{6^{r+s}\times 12^{r-s}}{8^r\times 9^{r+2s}}$$

is an integer if

○ $r+s\leq 0$
○ $s\leq 0$
○ $r\leq 0$
○ $r\geq s$

Solution: $s\leq 0$

1.2 QUADRATICS (SOLVING, COMPLETING SQUARE, GRAPHS, DISCRIMINANT)

Question 1 (STEP I 2013 Q1)

(i) Use the substitution $\sqrt{x} = y$ (where $y \geqslant 0$) to find the real root of the equation

$$x + 3\sqrt{x} - \tfrac{1}{2} = 0\,.$$

(ii) Find all real roots of the following equations:

(a) $x + 10\sqrt{x+2} \; - 22 = 0\,;$

(b) $x^2 - 4x + \sqrt{2x^2 - 8x - 3} \; - 9 = 0\,.$

Question 2 (STEP I 2009 Q3)

(i) By considering the equation $x^2 + x - a = 0\,$, show that the equation $x = (a - x)^{\frac{1}{2}}$ has one real solution when $a \geqslant 0$ and no real solutions when $a < 0\,$.

Find the number of distinct real solutions of the equation

$$x = \left((1 + a)x - a\right)^{\frac{1}{3}}$$

in the cases that arise according to the value of a.

(ii) Find the number of distinct real solutions of the equation

$$x = (b + x)^{\frac{1}{2}}$$

in the cases that arise according to the value of $b\,$.

Solution: (i) 1 real root if $a < -\frac{1}{4}$. 2 distinct real roots if $a = 2$

(ii) If $b = -\frac{1}{4}$, one solution. No solutions if $b < -\frac{1}{4}$. Two solutions if $-\frac{1}{4} < b \leq 0$

Question 3 (STEP I 2007 Q4)

Show that $x^3 - 3xbc + b^3 + c^3$ can be written in the form $(x + b + c)\,\mathrm{Q}(x)$, where $\mathrm{Q}(x)$ is a quadratic expression. Show that $2\mathrm{Q}(x)$ can be written as the sum of three expressions, each of which is a perfect square.

It is given that the equations $ay^2 + by + c = 0$ and $by^2 + cy + a = 0$ have a common root k. The coefficients a, b and c are real, a and b are both non-zero, and $ac \neq b^2$. Show that

$$\left(ac - b^2\right) k = bc - a^2$$

and determine a similar expression involving k^2. Hence show that

$$\left(ac - b^2\right)\left(ab - c^2\right) = \left(bc - a^2\right)^2$$

and that $a^3 - 3abc + b^3 + c^3 = 0\,$. Deduce that either $k = 1$ or the two equations are identical.

Question 4 (STEP I 2007 Q6)

(i) Given that $x^2 - y^2 = (x-y)^3$ and that $x - y = d$ (where $d \neq 0$), express each of x and y in terms of d. Hence find a pair of integers m and n satisfying $m - n = (\sqrt{m} - \sqrt{n})^3$ where $m > n > 100$.

(ii) Given that $x^3 - y^3 = (x-y)^4$ and that $x - y = d$ (where $d \neq 0$), show that $3xy = d^3 - d^2$. Hence show that

$$2x = d \pm d\sqrt{\frac{4d-1}{3}}$$

and determine a pair of distinct positive integers m and n such that $m^3 - n^3 = (m-n)^4$.

Solutions: (i) Example: $m = 441, n = 225$ (ii) Example: $m = 14, y = 7$

Question 5 (STEP I 2006 Q3)

In this question b, c, p and q are real numbers.

(i) By considering the graph $y = x^2 + bx + c$ show that $c < 0$ is a sufficient condition for the equation $x^2 + bx + c = 0$ to have distinct real roots. Determine whether $c < 0$ is a necessary condition for the equation to have distinct real roots.

(ii) Determine necessary and sufficient conditions for the equation $x^2 + bx + c = 0$ to have distinct positive real roots.

(iii) What can be deduced about the number and the nature of the roots of the equation $x^3 + px + q = 0$ if $p > 0$ and $q < 0$?

What can be deduced if $p < 0$ and $q < 0$? You should consider the different cases that arise according to the value of $4p^3 + 27q^2$.

Question 6 (STEP I 2005 Q3)

In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

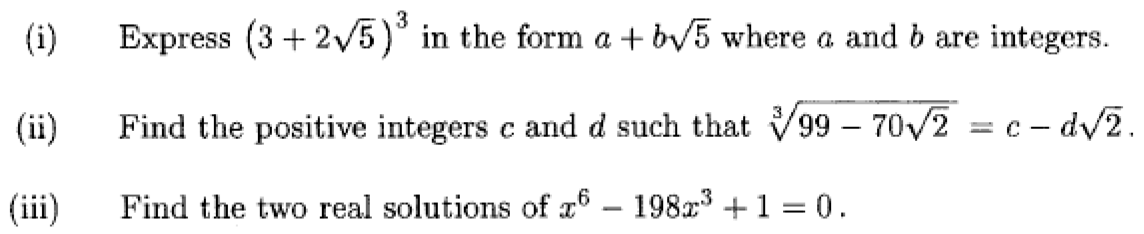
has two distinct real solutions.

(ii) Show that the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if $c^2 = -\frac{4ab}{(a-b)^2}$. Show that this condition can be written $c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$ and deduce that it can only hold if $0 < c^2 \leqslant 1$.

Question 7 (STEP I 2004 Q1)

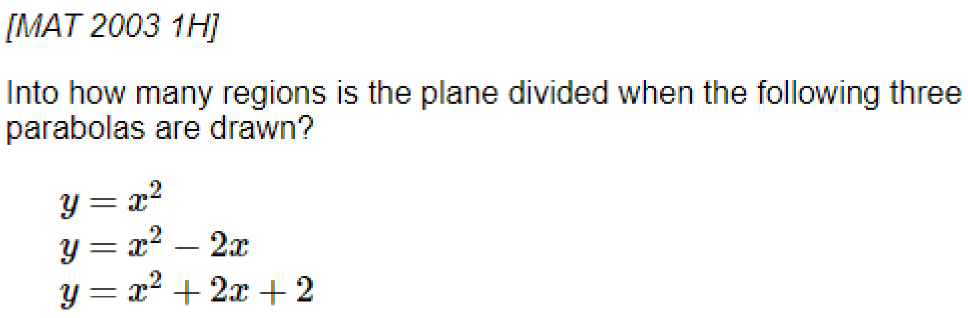


(i) Express $\left(3+2\sqrt{5}\right)^3$ in the form $a+b\sqrt{5}$ where a and b are integers.

(ii) Find the positive integers c and d such that $\sqrt[3]{99-70\sqrt{2}} = c - d\sqrt{2}$.

(iii) Find the two real solutions of $x^6 - 198x^3 + 1 = 0$.

Question 8



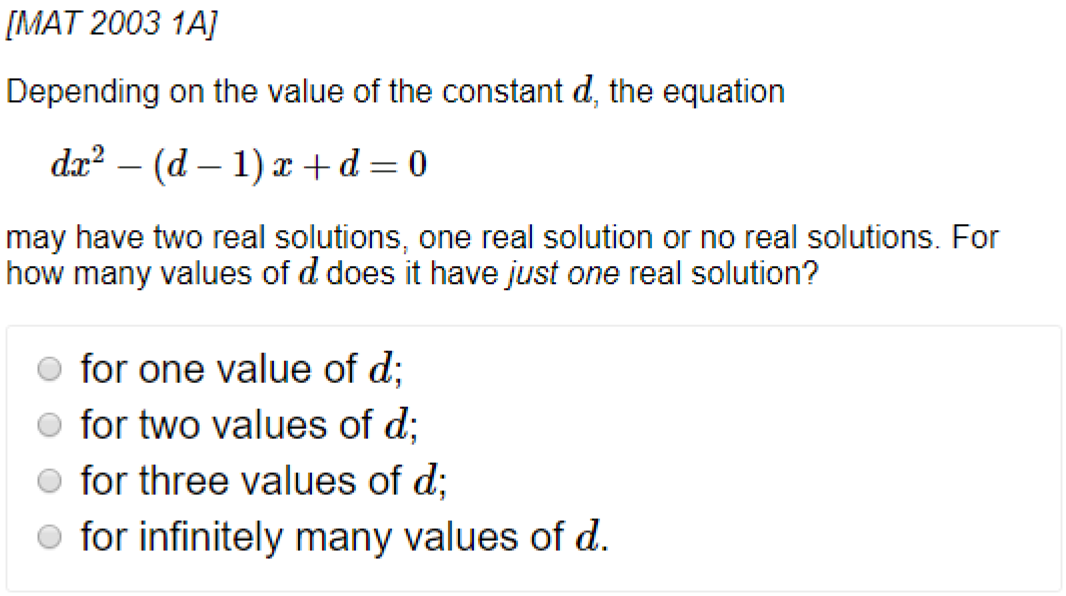
[MAT 2003 1H]

Into how many regions is the plane divided when the following three parabolas are drawn?

$$y = x^2$$
$$y = x^2 - 2x$$
$$y = x^2 + 2x + 2$$

Solution: 7

Question 9



[MAT 2003 1A]

Depending on the value of the constant d, the equation

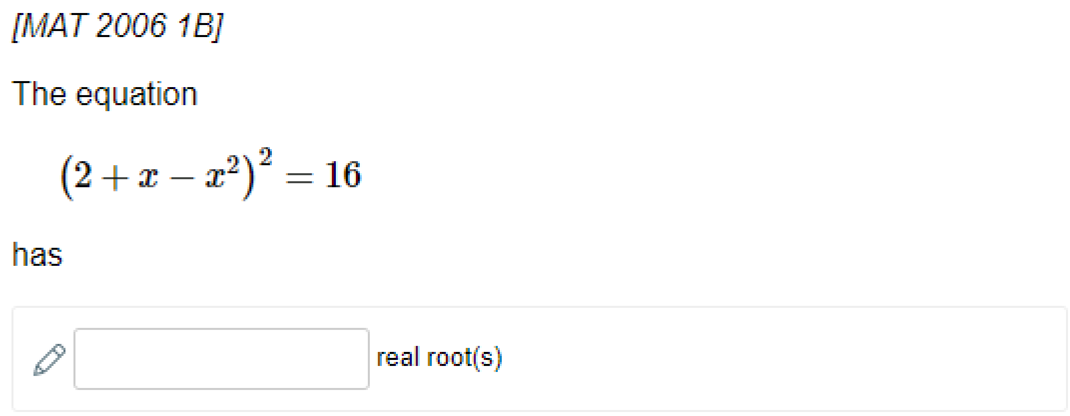
$$dx^2 - (d-1)\,x + d = 0$$

may have two real solutions, one real solution or no real solutions. For how many values of d does it have *just one* real solution?

○ for one value of d;
○ for two values of d;
○ for three values of d;
○ for infinitely many values of d.

Solution: Option 2

Question 10



[MAT 2006 1B]

The equation

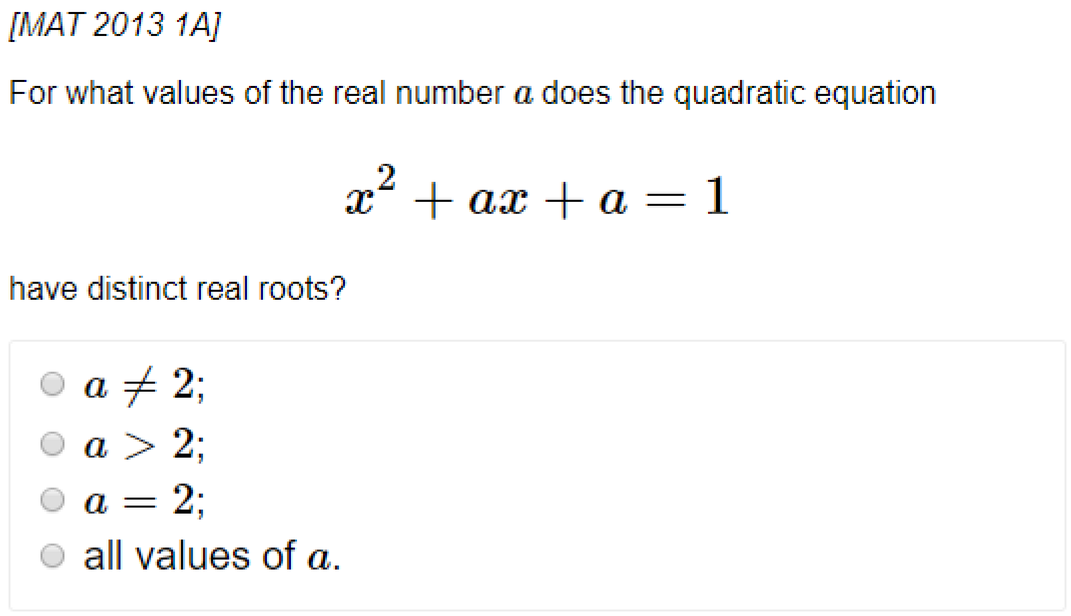
$$\left(2+x-x^2\right)^2=16$$

has

☐ real root(s)

Solution: 2 real roots

Question 11



[MAT 2013 1A]

For what values of the real number a does the quadratic equation

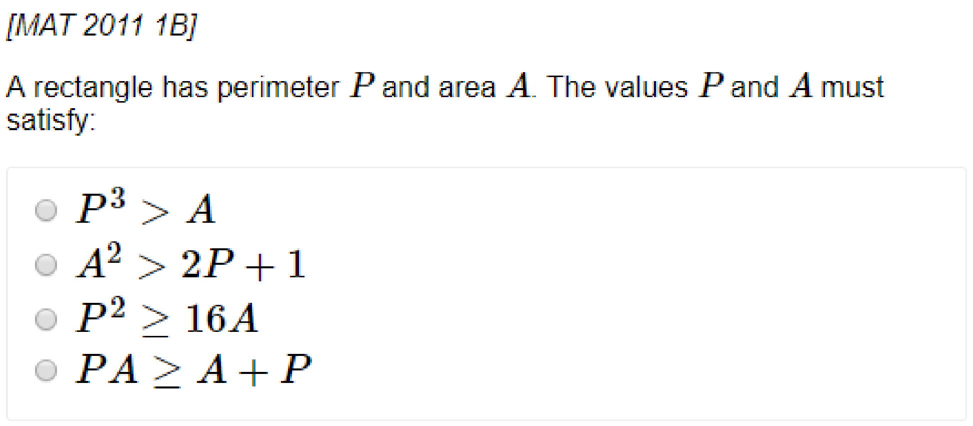
$$x^2+ax+a=1$$

have distinct real roots?

- ○ $a \neq 2$;
- ○ $a > 2$;
- ○ $a = 2$;
- ○ all values of a.

Solution: $a \neq 0$

Question 12



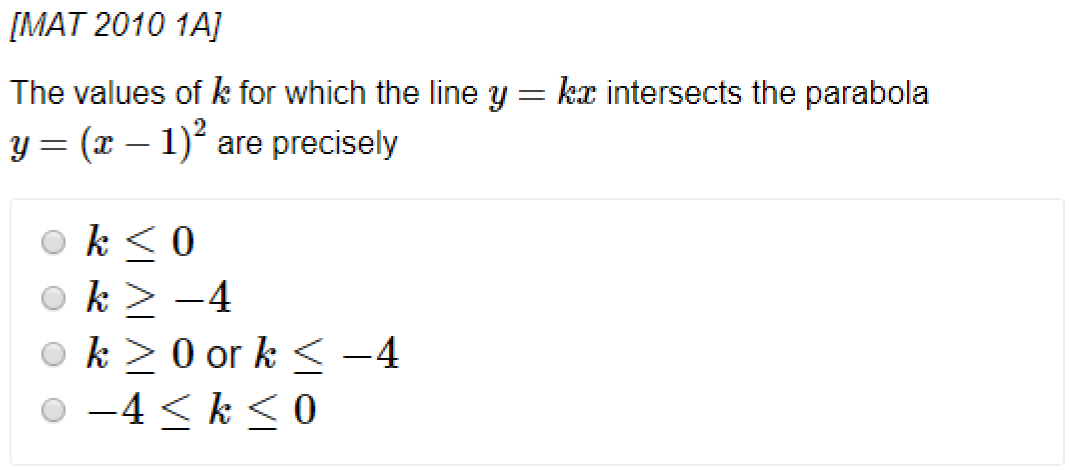
[MAT 2011 1B]

A rectangle has perimeter P and area A. The values P and A must satisfy:

- ○ $P^3 > A$
- ○ $A^2 > 2P+1$
- ○ $P^2 \geq 16A$
- ○ $PA \geq A+P$

Solution: Option 3

Question 13



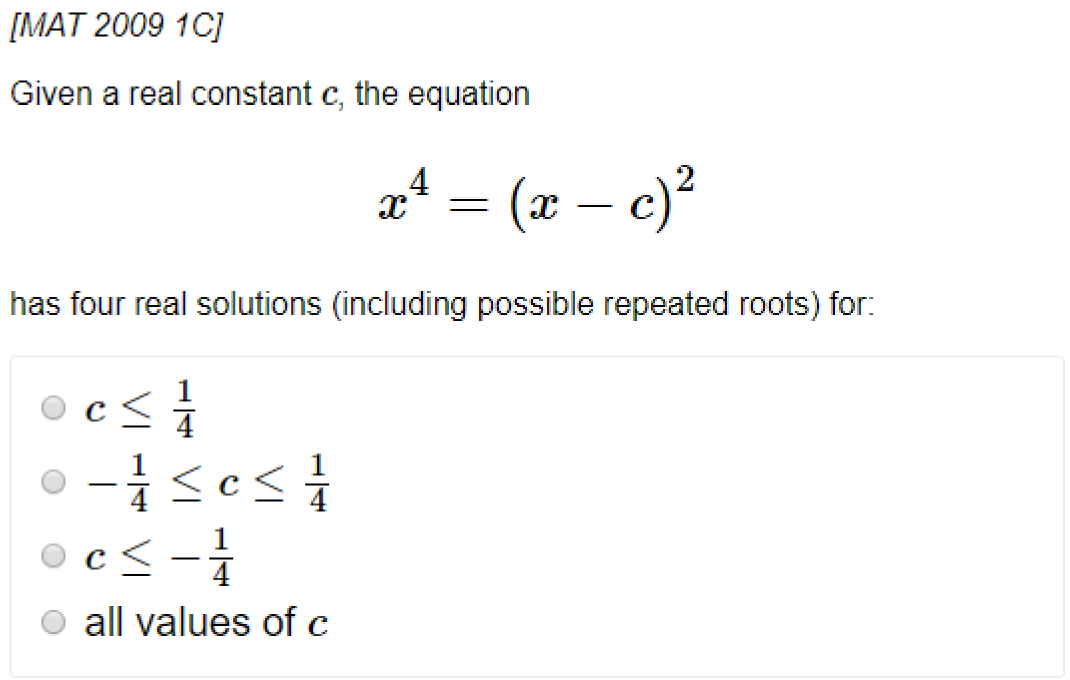
[MAT 2010 1A]

The values of k for which the line $y = kx$ intersects the parabola $y = (x-1)^2$ are precisely

○ $k \leq 0$
○ $k \geq -4$
○ $k \geq 0$ or $k \leq -4$
○ $-4 \leq k \leq 0$

Solution: Option 3

Question 14



[MAT 2009 1C]

Given a real constant c, the equation

$$x^4 = (x-c)^2$$

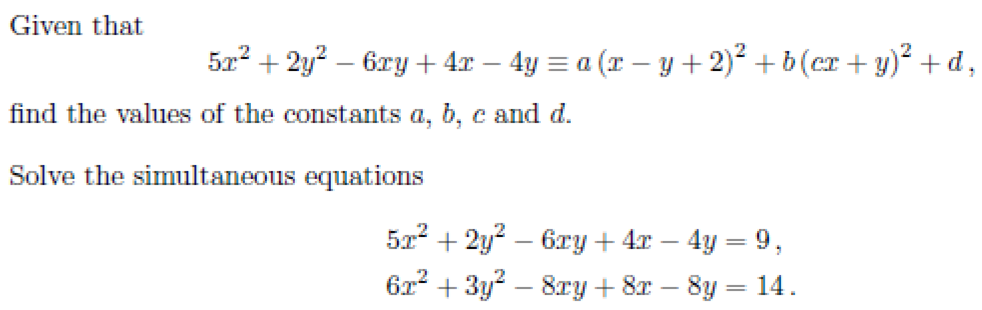
has four real solutions (including possible repeated roots) for:

○ $c \leq \frac{1}{4}$
○ $-\frac{1}{4} \leq c \leq \frac{1}{4}$
○ $c \leq -\frac{1}{4}$
○ all values of c

Solution: Option 2

1.3 EQUATIONS & INEQUALITIES (SIMULTANEOUS EQNS, QUADRATIC INEQUALITIES)

Question 1 (STEP 2010 Q1)



Given that

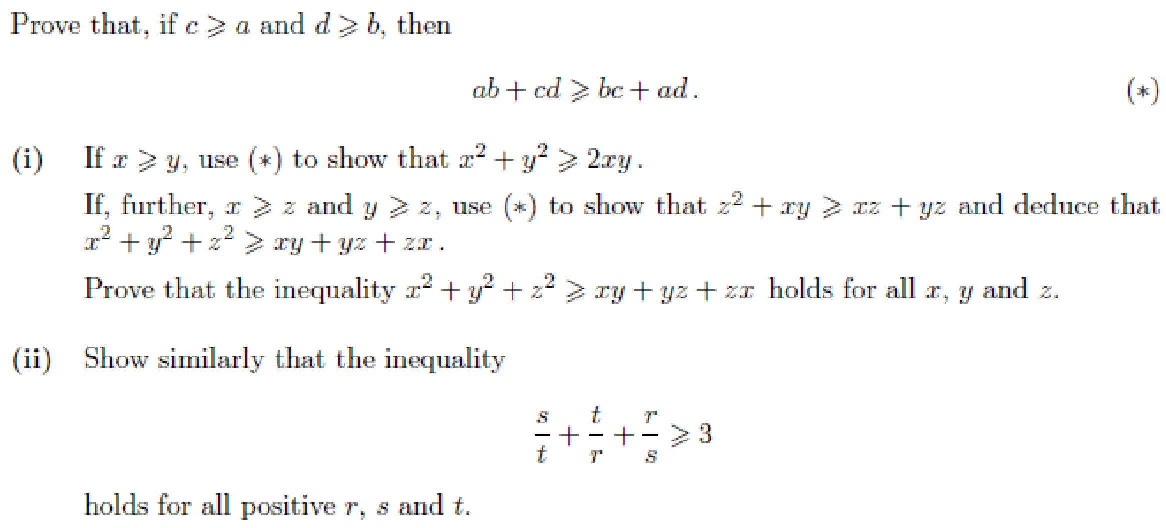
$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a\,(x - y + 2)^2 + b\,(cx + y)^2 + d\,,$$

find the values of the constants a, b, c and d.

Solve the simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9\,,$$
$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14\,.$$

Question 2 (STEP 2008 Q3)



Prove that, if $c \geqslant a$ and $d \geqslant b$, then

$$ab + cd \geqslant bc + ad\,. \qquad (*)$$

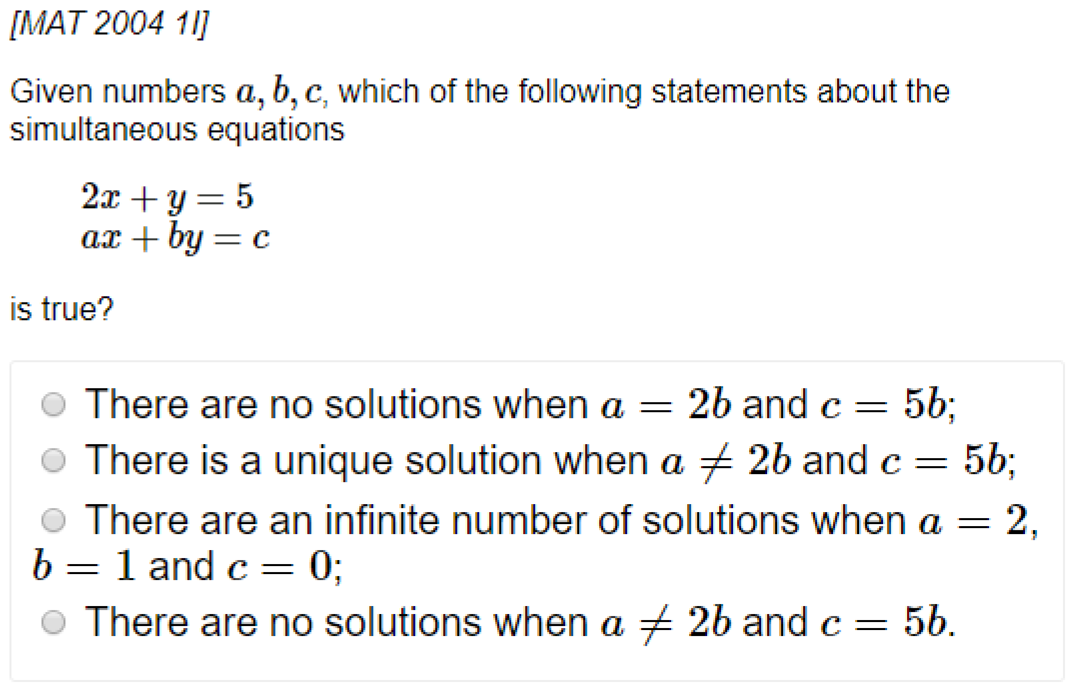
(i) If $x \geqslant y$, use $(*)$ to show that $x^2 + y^2 \geqslant 2xy\,$.
If, further, $x \geqslant z$ and $y \geqslant z$, use $(*)$ to show that $z^2 + xy \geqslant xz + yz$ and deduce that $x^2 + y^2 + z^2 \geqslant xy + yz + zx\,$.
Prove that the inequality $x^2 + y^2 + z^2 \geqslant xy + yz + zx$ holds for all x, y and z.

(ii) Show similarly that the inequality

$$\frac{s}{t} + \frac{t}{r} + \frac{r}{s} \geqslant 3$$

holds for all positive r, s and t.

Question 3



[MAT 2004 1I]

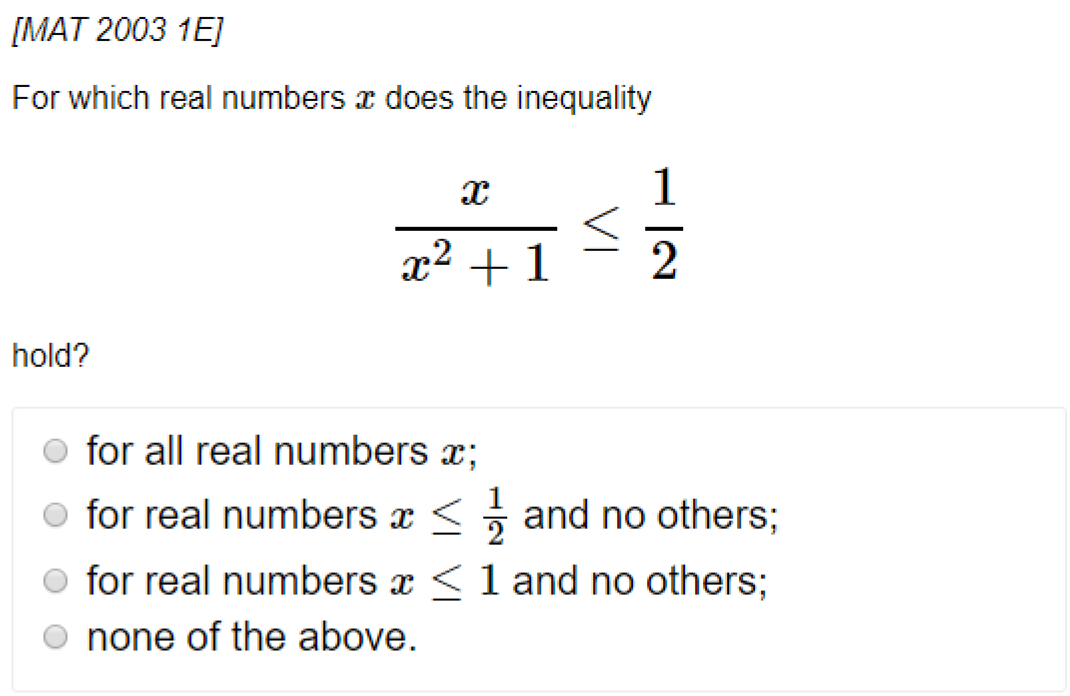
Given numbers a, b, c, which of the following statements about the simultaneous equations

$$2x + y = 5$$
$$ax + by = c$$

is true?

- ○ There are no solutions when $a = 2b$ and $c = 5b$;
- ○ There is a unique solution when $a \neq 2b$ and $c = 5b$;
- ○ There are an infinite number of solutions when $a = 2$, $b = 1$ and $c = 0$;
- ○ There are no solutions when $a \neq 2b$ and $c = 5b$.

Question 4



[MAT 2003 1E]

For which real numbers x does the inequality

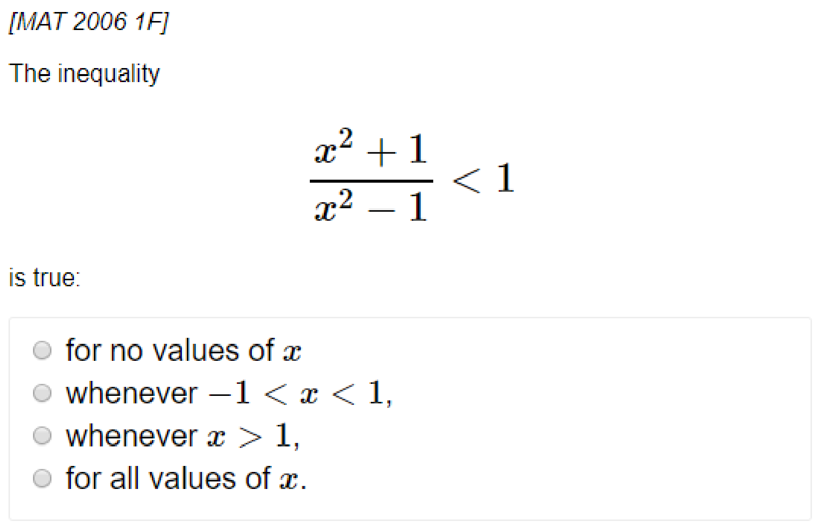
$$\frac{x}{x^2+1} \leq \frac{1}{2}$$

hold?

- ○ for all real numbers x;
- ○ for real numbers $x \leq \frac{1}{2}$ and no others;
- ○ for real numbers $x \leq 1$ and no others;
- ○ none of the above.

Solution: Option 1

Question 5



[MAT 2006 1F]

The inequality

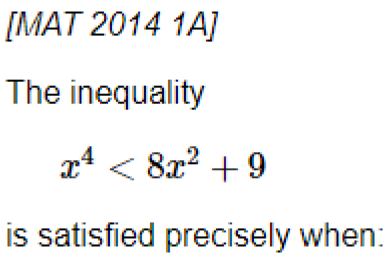
$$\frac{x^2+1}{x^2-1} < 1$$

is true:

- ○ for no values of x
- ○ whenever $-1 < x < 1$,
- ○ whenever $x > 1$,
- ○ for all values of x.

Solution: Option 2

Question 6



[MAT 2014 1A]

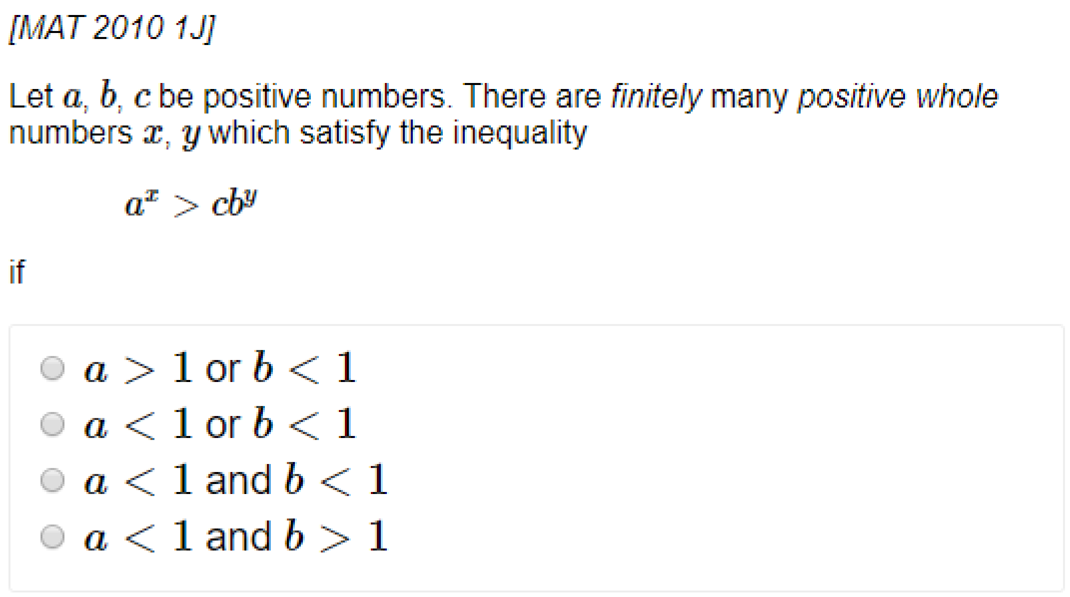
The inequality

$$x^4 < 8x^2 + 9$$

is satisfied precisely when:

Solution: $-3 < x < 3$

Question 7



[MAT 2010 1J]

Let a, b, c be positive numbers. There are *finitely* many *positive whole* numbers x, y which satisfy the inequality

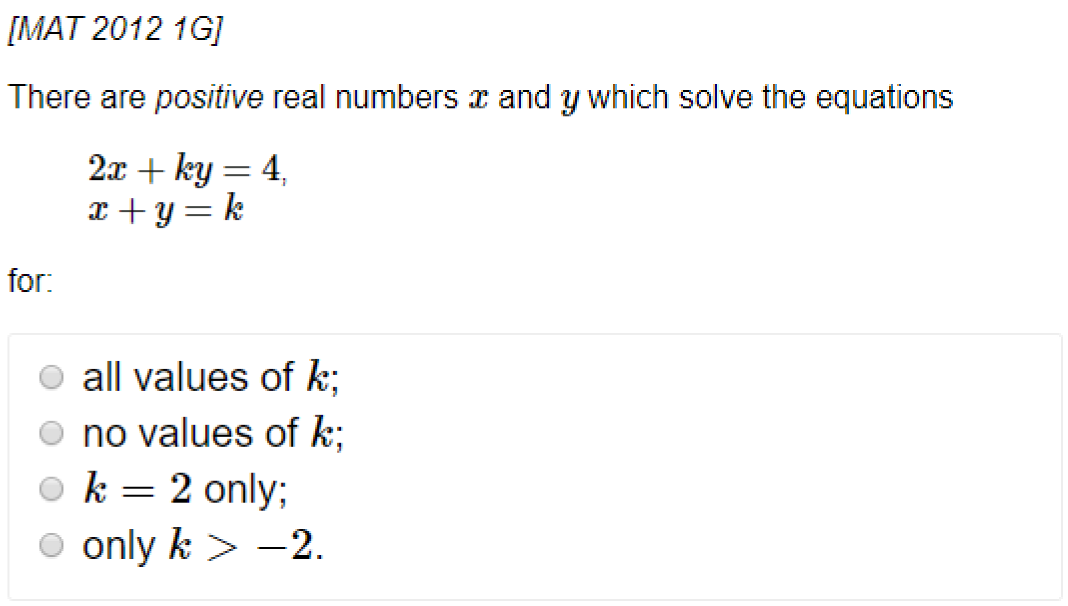
$$a^x > cb^y$$

if

- ○ $a > 1$ or $b < 1$
- ○ $a < 1$ or $b < 1$
- ○ $a < 1$ and $b < 1$
- ○ $a < 1$ and $b > 1$

Solution: Option 4

Question 8



[MAT 2012 1G]

There are *positive* real numbers x and y which solve the equations

$$2x + ky = 4,$$
$$x + y = k$$

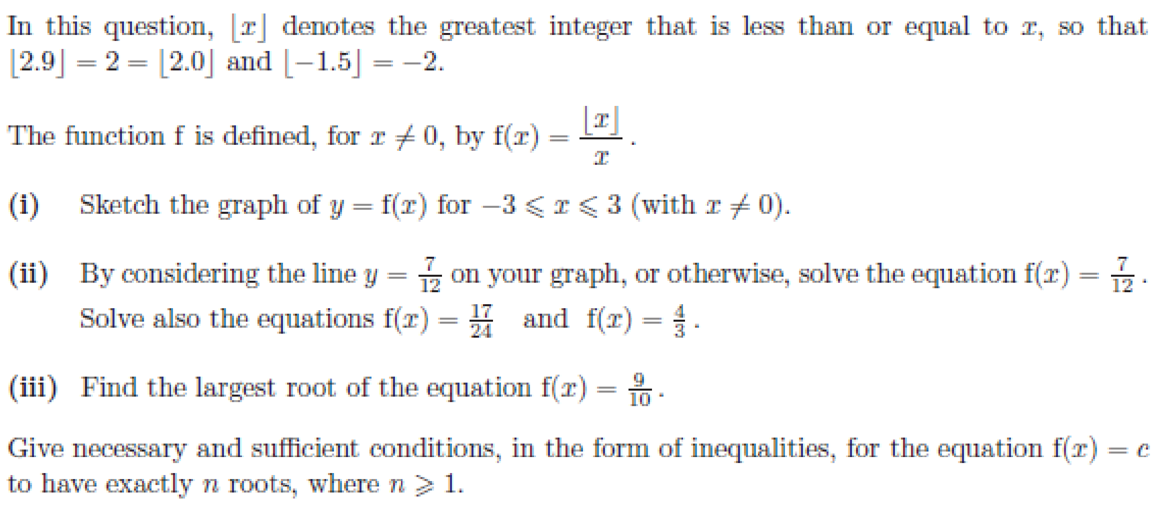
for:

- ○ all values of k;
- ○ no values of k;
- ○ $k = 2$ only;
- ○ only $k > -2$.

Solution: Option 3

1.4 GRAPHS & TRANSFORMATIONS (CUBICS, QUARTICS, RECIPROCAL, TRANSFORMING, POINTS OF INTERSECTION)

Question 1 (STEP I 2013 Q2)



In this question, $\lfloor x \rfloor$ denotes the greatest integer that is less than or equal to x, so that $\lfloor 2.9 \rfloor = 2 = \lfloor 2.0 \rfloor$ and $\lfloor -1.5 \rfloor = -2$.

The function f is defined, for $x \neq 0$, by $\mathrm{f}(x) = \dfrac{\lfloor x \rfloor}{x}$.

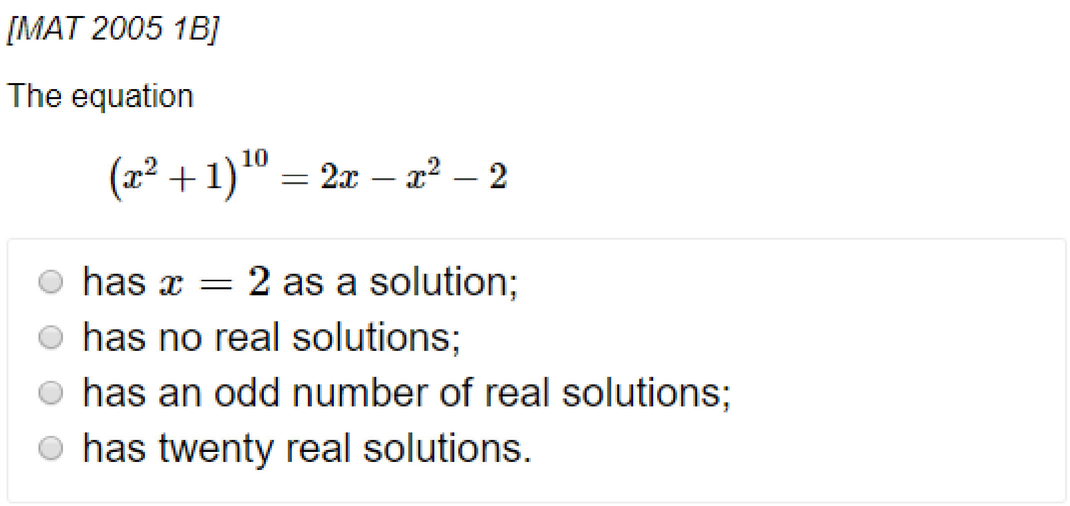
(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-3 \leqslant x \leqslant 3$ (with $x \neq 0$).

(ii) By considering the line $y = \frac{7}{12}$ on your graph, or otherwise, solve the equation $\mathrm{f}(x) = \frac{7}{12}$. Solve also the equations $\mathrm{f}(x) = \frac{17}{24}$ and $\mathrm{f}(x) = \frac{4}{3}$.

(iii) Find the largest root of the equation $\mathrm{f}(x) = \frac{9}{10}$.

Give necessary and sufficient conditions, in the form of inequalities, for the equation $\mathrm{f}(x) = c$ to have exactly n roots, where $n \geqslant 1$.

Question 2



[MAT 2005 1B]

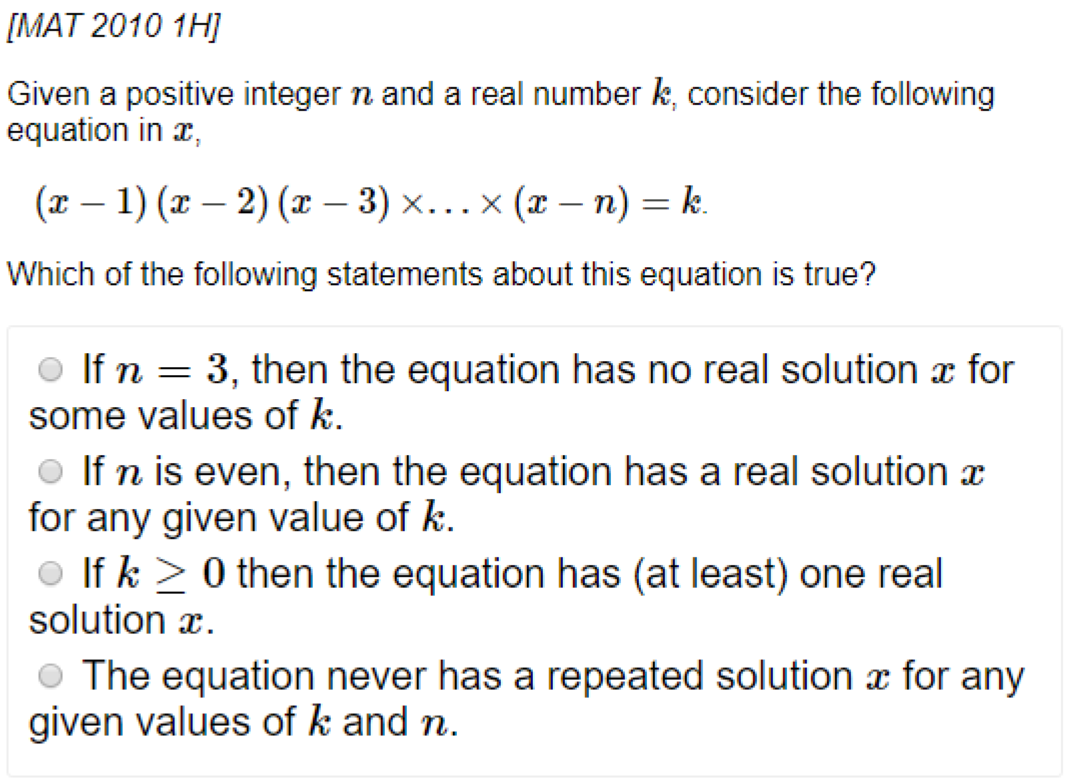
The equation

$$\left(x^2+1\right)^{10} = 2x - x^2 - 2$$

- ○ has $x = 2$ as a solution;
- ○ has no real solutions;
- ○ has an odd number of real solutions;
- ○ has twenty real solutions.

Solution: Option 2

Question 3



[MAT 2010 1H]

Given a positive integer n and a real number k, consider the following equation in x,

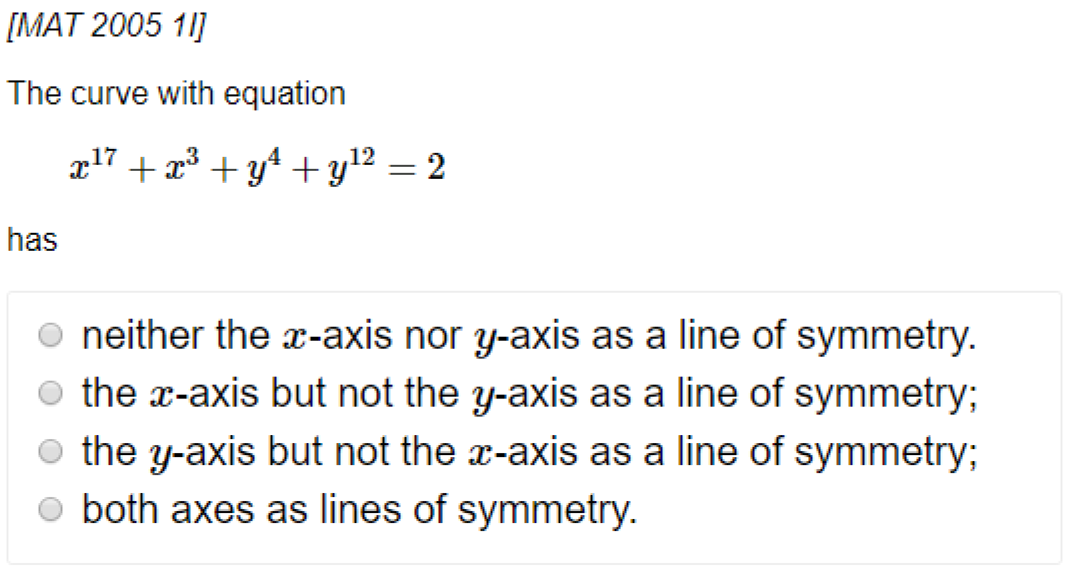
$$(x-1)(x-2)(x-3)\times\ldots\times(x-n)=k.$$

Which of the following statements about this equation is true?

- ○ If $n=3$, then the equation has no real solution x for some values of k.
- ○ If n is even, then the equation has a real solution x for any given value of k.
- ○ If $k\geq 0$ then the equation has (at least) one real solution x.
- ○ The equation never has a repeated solution x for any given values of k and n.

Solution: Option 3

Question 4



[MAT 2005 1I]

The curve with equation

$$x^{17}+x^{3}+y^{4}+y^{12}=2$$

has

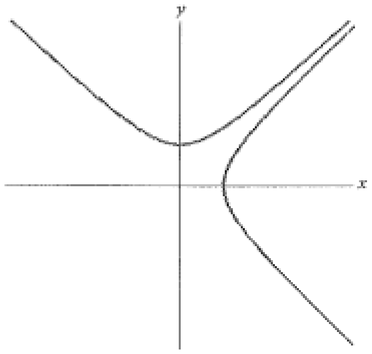
- ○ neither the x-axis nor y-axis as a line of symmetry.
- ○ the x-axis but not the y-axis as a line of symmetry;
- ○ the y-axis but not the x-axis as a line of symmetry;
- ○ both axes as lines of symmetry.

Solution: Option 2

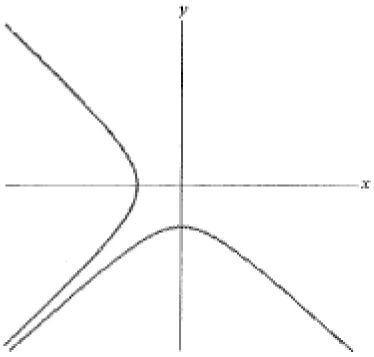
Question 5

[MAT 2004 1H]

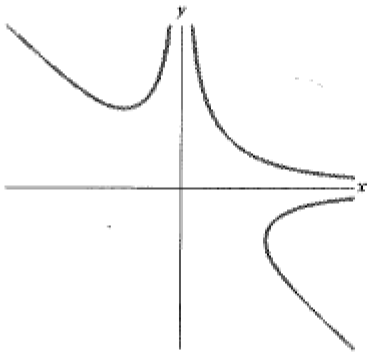
A sketch of the curve with equation $x^2y^2\left(x+y\right)=1$ is drawn in



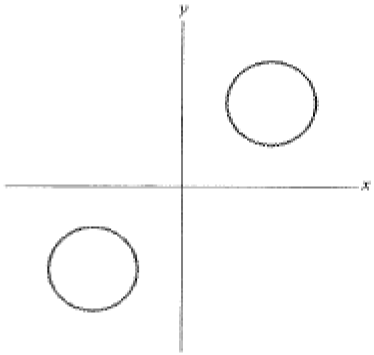
(a)



(b)



(c)



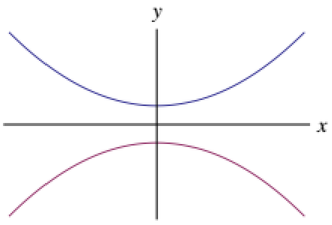
(d)

Solution: (c)

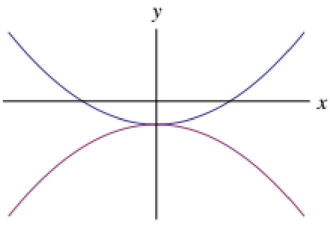
Question 6

[MAT 2013 1D]

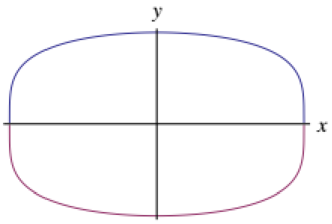
Which of the following sketches is a graph of $x^4-y^2=2y+1$



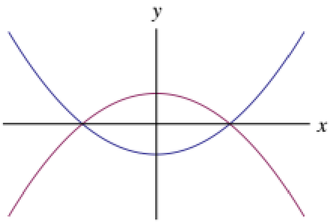
(a)



(b)



(c)



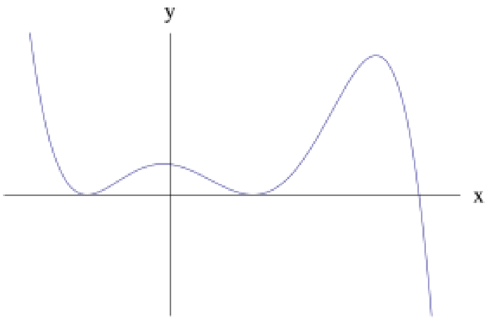
(d)

Solution: (b)

Question 7

[MAT 2012 1E]

Which one of the following equations could possibly could possibly have the graph given below?



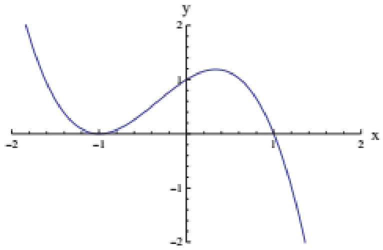
- ○ $y = (3-x)^2(3+x)^2(1-x)$
- ○ $y = -x^2(x-9)(x^2-3)$
- ○ $y = (x-6)(x-2)^2(x+2)^2$
- ○ $y = (x^2-1)^2(3-x)$

Solution: Option 4

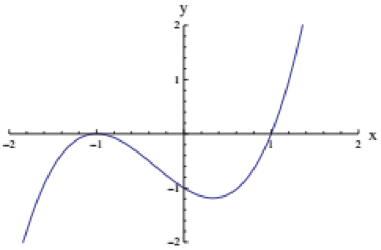
Question 8

[MAT 2011 1A]

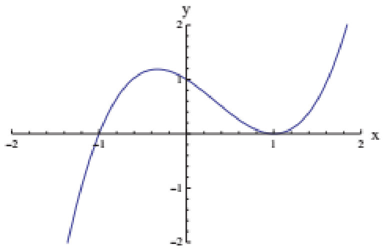
A sketch of the graph $y = x^3 - x^2 - x + 1$ appears on which of the following axes?



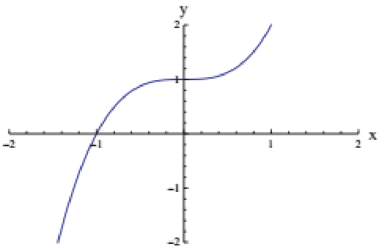
(a)



(b)



(c)



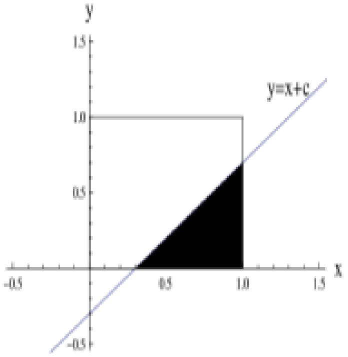
(d)

Solution: (c)

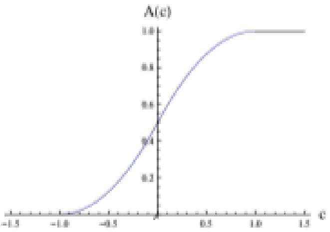
Question 9

[MAT 2012 1D]

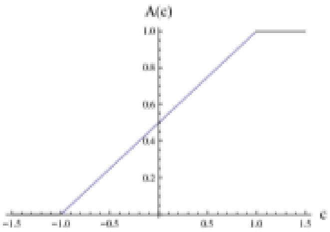
Shown below is a diagram of the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$ and the line $y=x+c$. The shaded region is the region of the square which lies below the line; this shaded region has area $A(c)$.



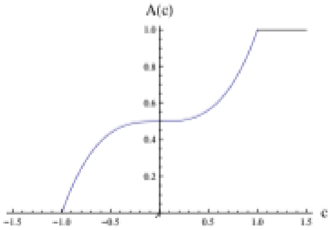
Which of the following graphs shows $A(c)$ as c varies?



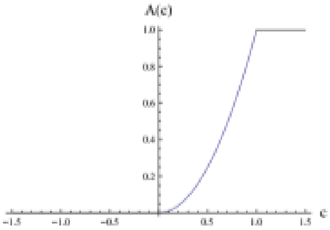
(a)



(b)



(c)

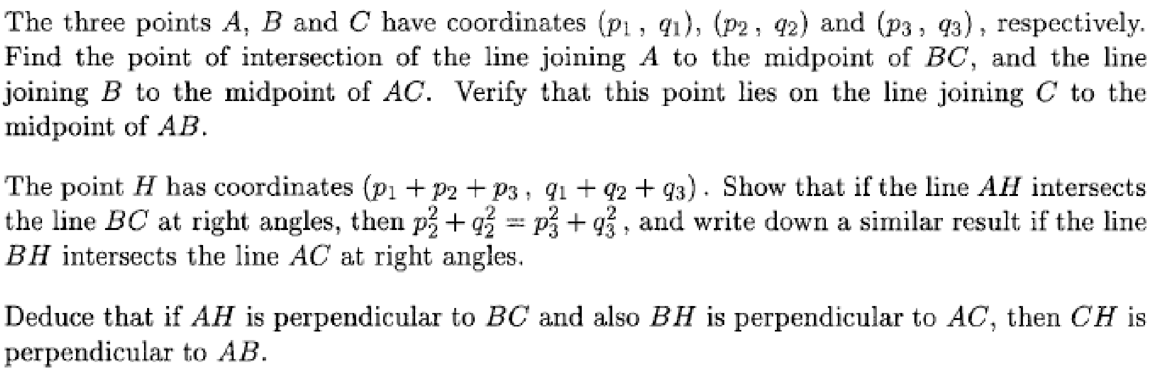


(d)

Solution: (a)

1.5 STRAIGHT LINE GRAPHS

Question 1 (STEP I 2004 Q6)

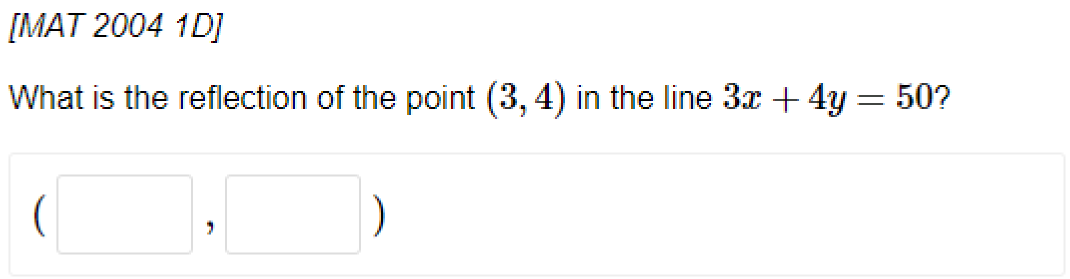


The three points A, B and C have coordinates $(p_1\,,\ q_1)$, $(p_2\,,\ q_2)$ and $(p_3\,,\ q_3)$, respectively. Find the point of intersection of the line joining A to the midpoint of BC, and the line joining B to the midpoint of AC. Verify that this point lies on the line joining C to the midpoint of AB.

The point H has coordinates $(p_1+p_2+p_3\,,\ q_1+q_2+q_3)$. Show that if the line AH intersects the line BC at right angles, then $p_2^2+q_2^2=p_3^2+q_3^2$, and write down a similar result if the line BH intersects the line AC at right angles.

Deduce that if AH is perpendicular to BC and also BH is perpendicular to AC, then CH is perpendicular to AB.

Question 2



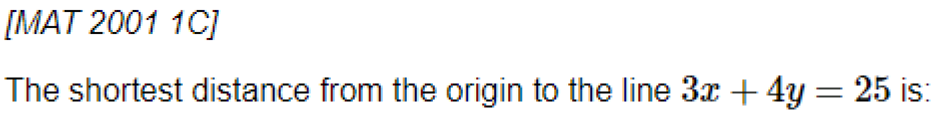
[MAT 2004 1D]

What is the reflection of the point $(3,4)$ in the line $3x+4y=50$?

(☐ , ☐)

Solution: (9,12)

Question 3

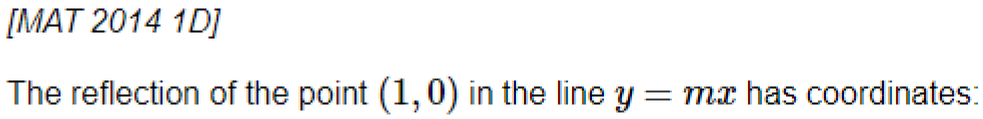


[MAT 2001 1C]

The shortest distance from the origin to the line $3x+4y=25$ is:

Solution: 5

Question 4



[MAT 2014 1D]

The reflection of the point $(1,0)$ in the line $y=mx$ has coordinates:

Solution: $\left(\frac{1-m^2}{1+m^2},\frac{2m}{1+m^2}\right)$

1.6 EQUATIONS OF CIRCLES

Question 1 (STEP I 2013 Q5)

The point P has coordinates (x, y) which satisfy

$$x^2 + y^2 + kxy + 3x + y = 0\,.$$

(i) Sketch the locus of P in the case $k = 0$, giving the points of intersection with the coordinate axes.

(ii) By factorising $3x^2 + 3y^2 + 10xy$, or otherwise, sketch the locus of P in the case $k = \frac{10}{3}$, giving the points of intersection with the coordinate axes.

(iii) In the case $k = 2$, let Q be the point obtained by rotating P clockwise about the origin by an angle θ, so that the coordinates (X, Y) of Q are given by

$$X = x\cos\theta + y\sin\theta\,, \qquad Y = -x\sin\theta + y\cos\theta\,.$$

Show that, for $\theta = 45°$, the locus of Q is $\sqrt{2}Y = (\sqrt{2}X + 1)^2 - 1$.

Hence, or otherwise, sketch the locus of P in the case $k = 2$, giving the equation of the line of symmetry.

Question 2 (STEP 2009 Q8)

(i) The equation of the circle C is

$$(x - 2t)^2 + (y - t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y = 0$.

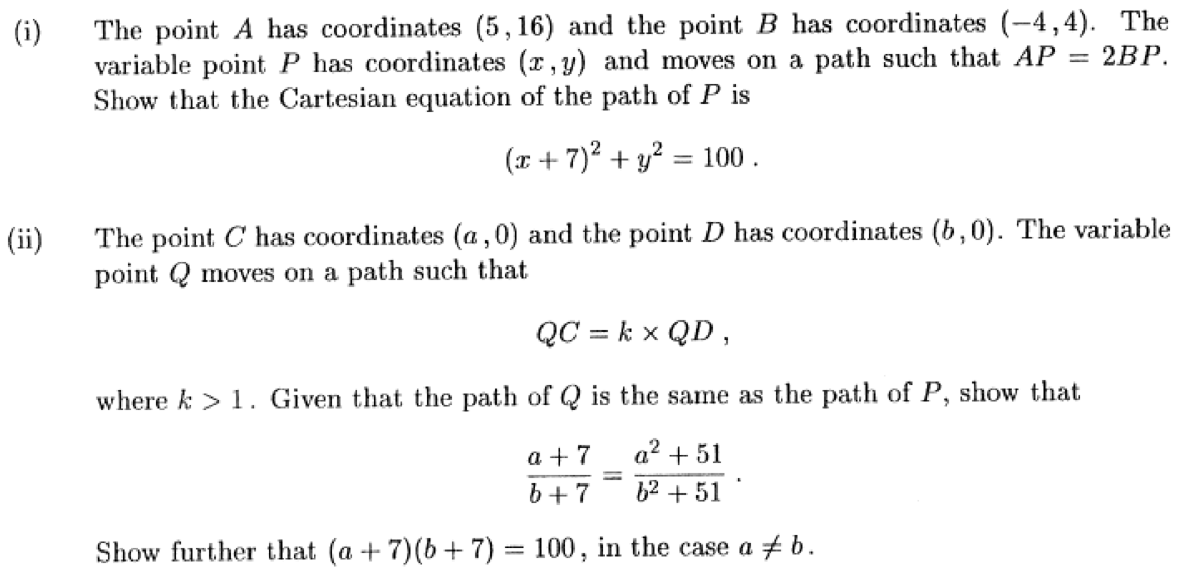
~~Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that~~ C touches the line $3y = 4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Solution: (ii) $(x - 2)^2 + (y - 1)^2 = 1$

Question 3 (STEP 2005 Q6)



(i) The point A has coordinates $(5\,,16)$ and the point B has coordinates $(-4\,,4)$. The variable point P has coordinates $(x\,,y)$ and moves on a path such that $AP = 2BP$. Show that the Cartesian equation of the path of P is

$$(x+7)^2 + y^2 = 100\ .$$

(ii) The point C has coordinates $(a\,,0)$ and the point D has coordinates $(b\,,0)$. The variable point Q moves on a path such that

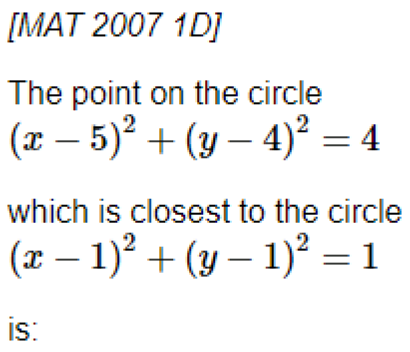
$$QC = k \times QD\ ,$$

where $k > 1$. Given that the path of Q is the same as the path of P, show that

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}\ .$$

Show further that $(a+7)(b+7) = 100$, in the case $a \neq b$.

Question 4



[MAT 2007 1D]

The point on the circle

$(x-5)^2 + (y-4)^2 = 4$

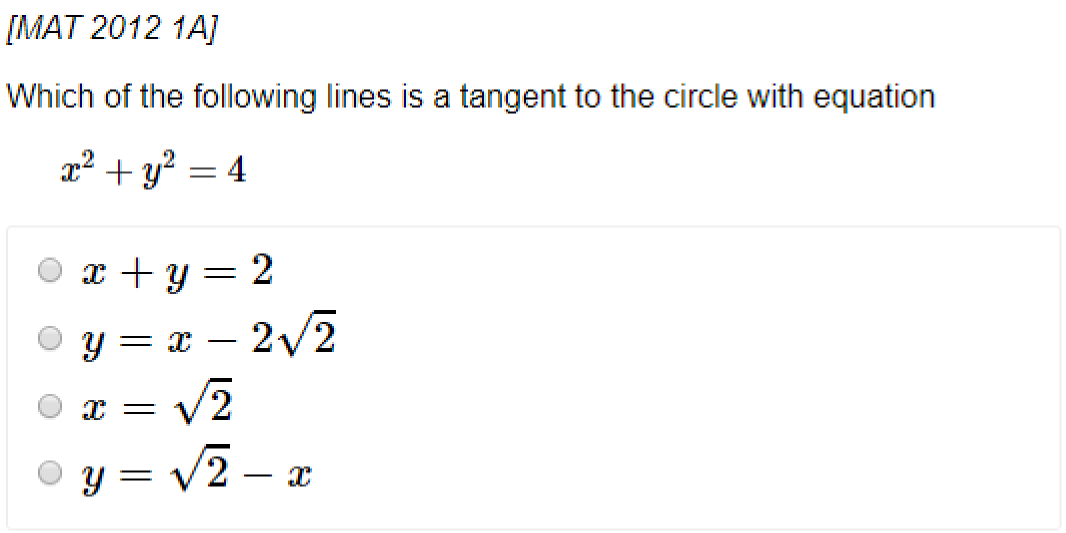
which is closest to the circle

$(x-1)^2 + (y-1)^2 = 1$

is:

Solution (3.4, 2.8)

Question 5



[MAT 2012 1A]

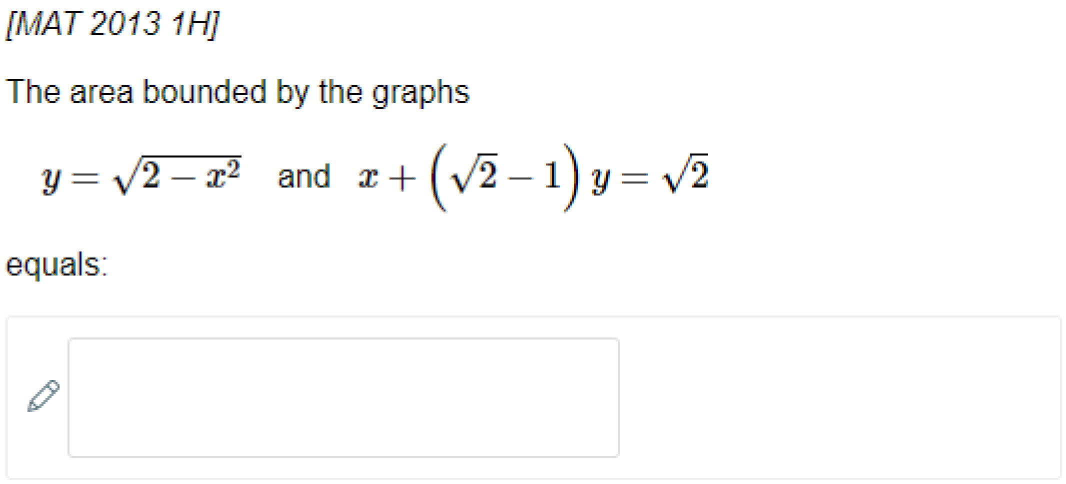
Which of the following lines is a tangent to the circle with equation

$x^2 + y^2 = 4$

○ $x + y = 2$
○ $y = x - 2\sqrt{2}$
○ $x = \sqrt{2}$
○ $y = \sqrt{2} - x$

Solution: Option 2

Question 6



[MAT 2013 1H]

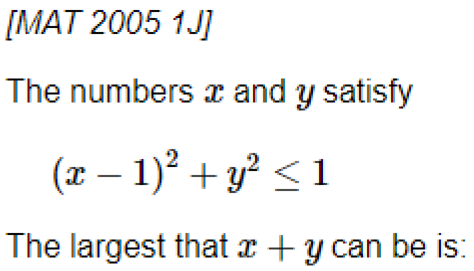
The area bounded by the graphs

$$y=\sqrt{2-x^2} \quad \text{and} \quad x+\left(\sqrt{2}-1\right)y=\sqrt{2}$$

equals:

Solution: $\frac{\pi}{4}-\frac{1}{\sqrt{2}}$

Question 7



[MAT 2005 1J]

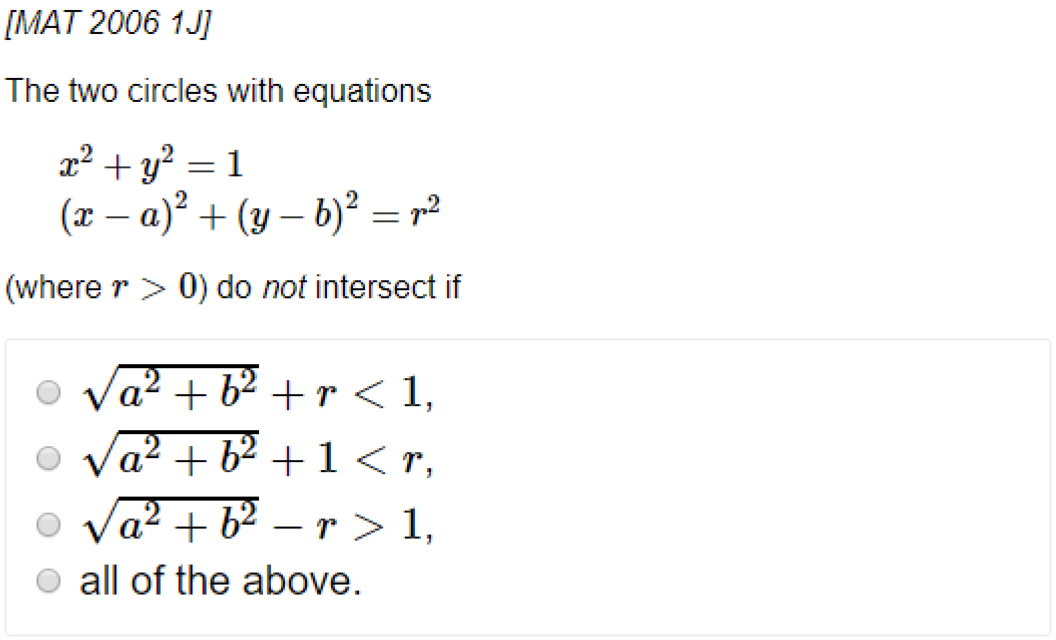
The numbers x and y satisfy

$$(x-1)^2+y^2\leq 1$$

The largest that $x+y$ can be is:

Solution: $1+\sqrt{2}$

Question 8



[MAT 2006 1J]

The two circles with equations

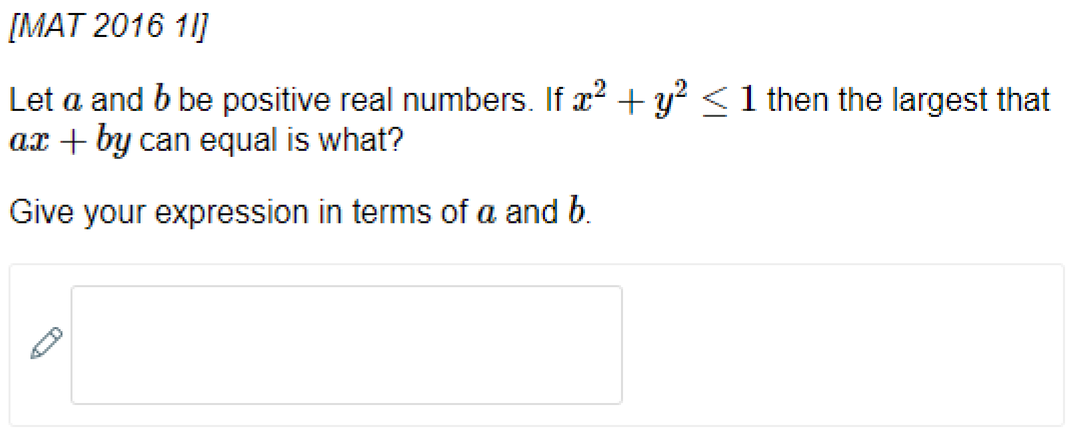
$$x^2+y^2=1$$
$$(x-a)^2+(y-b)^2=r^2$$

(where $r>0$) do *not* intersect if

- ○ $\sqrt{a^2+b^2}+r<1,$
- ○ $\sqrt{a^2+b^2}+1<r,$
- ○ $\sqrt{a^2+b^2}-r>1,$
- ○ all of the above.

Solution: Option 3.

Question 9



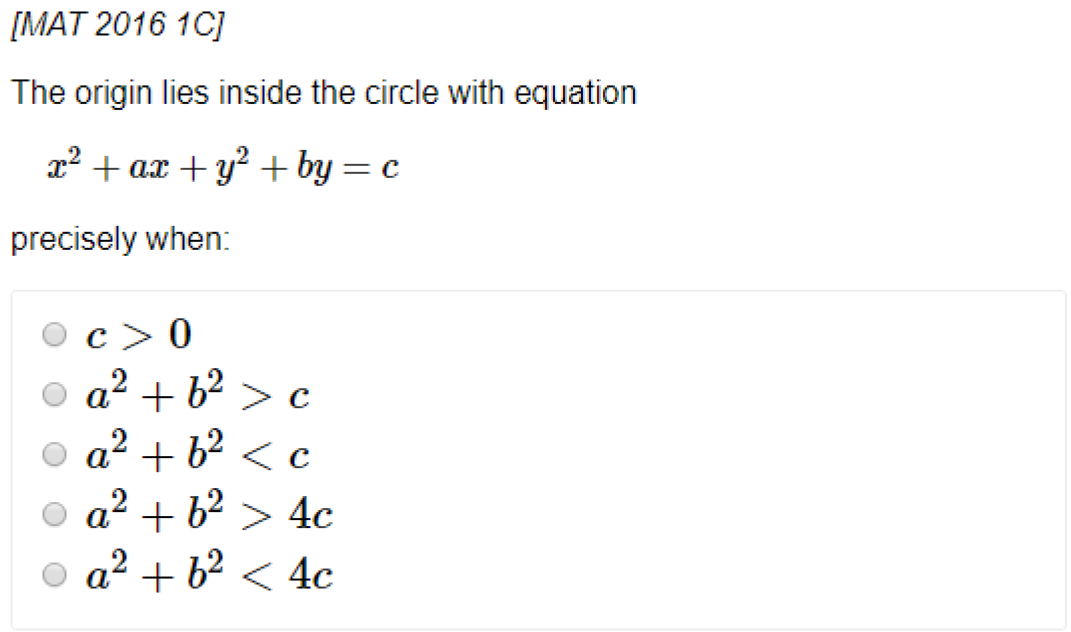
[MAT 2016 1I]

Let a and b be positive real numbers. If $x^2 + y^2 \leq 1$ then the largest that $ax + by$ can equal is what?

Give your expression in terms of a and b.

Solution: $\sqrt{a^2 + b^2}$

Question 10



[MAT 2016 1C]

The origin lies inside the circle with equation

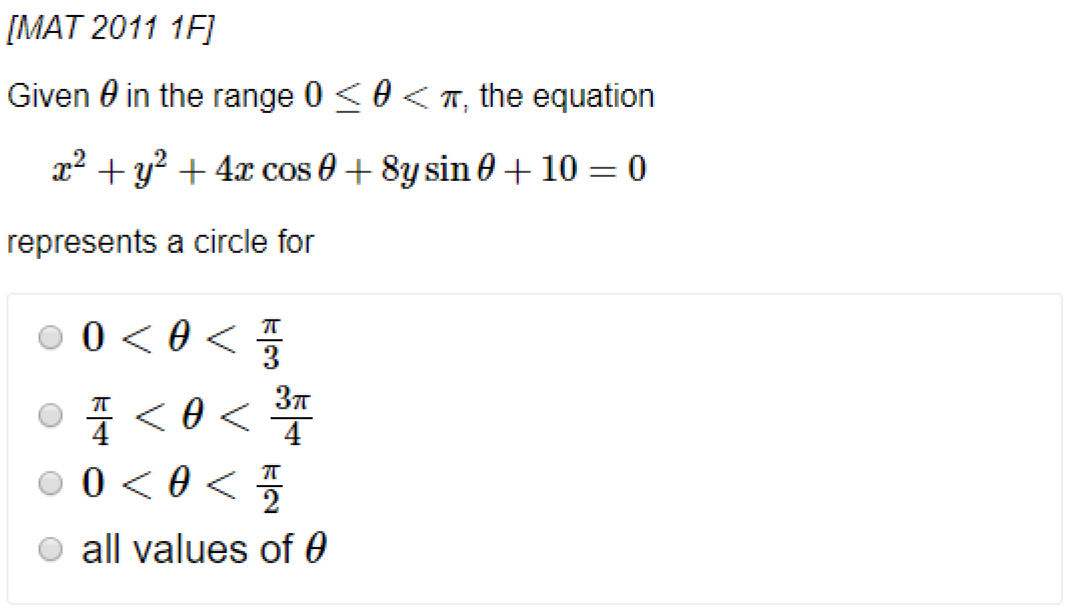
$$x^2 + ax + y^2 + by = c$$

precisely when:

- ○ $c > 0$
- ○ $a^2 + b^2 > c$
- ○ $a^2 + b^2 < c$
- ○ $a^2 + b^2 > 4c$
- ○ $a^2 + b^2 < 4c$

Solution: $c > 0$

Question 11



[MAT 2011 1F]

Given θ in the range $0 \leq \theta < \pi$, the equation

$$x^2 + y^2 + 4x\cos\theta + 8y\sin\theta + 10 = 0$$

represents a circle for

○ $0 < \theta < \frac{\pi}{3}$

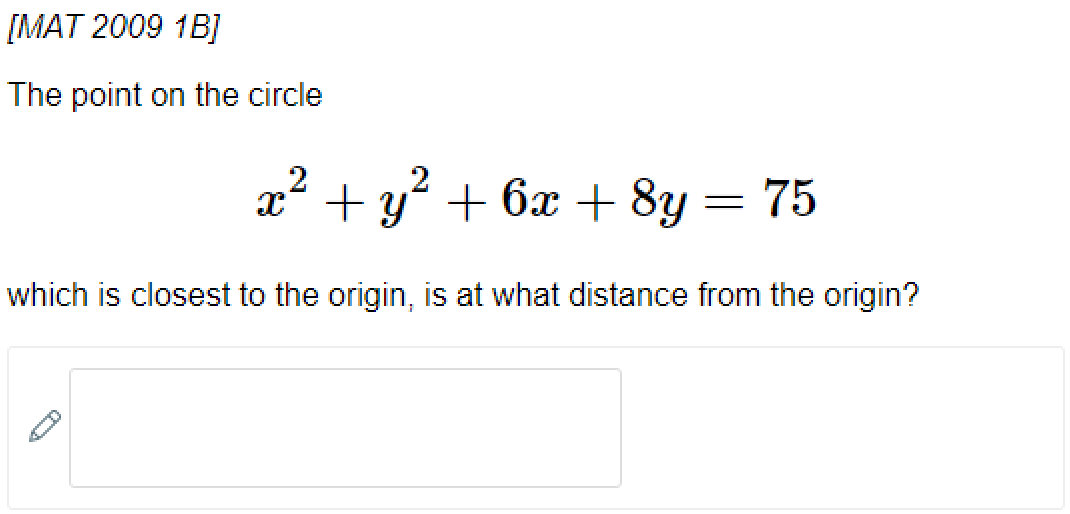
○ $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$

○ $0 < \theta < \frac{\pi}{2}$

○ all values of θ

Solution: Option 2

Question 12



[MAT 2009 1B]

The point on the circle

$$x^2 + y^2 + 6x + 8y = 75$$

which is closest to the origin, is at what distance from the origin?

Solution: 5

Question 13 (AEA 2006 Q4)

The line with equation $y = mx$ is a tangent to the circle C_1 with equation

$$(x+4)^2 + (y-7)^2 = 13.$$

(*a*) Show that m satisfies the equation

$$3m^2 + 56m + 36 = 0.$$

(4)

The tangents from the origin O to C_1 touch C_1 at the points A and B.

(*b*) Find the coordinates of the points A and B.

(8)

Another circle C_2 has equation $x^2 + y^2 = 13$. The tangents from the point $(4, -7)$ to C_2 touch it at the points P and Q.

(*c*) Find the coordinates of either the point P or the point Q.

(2)

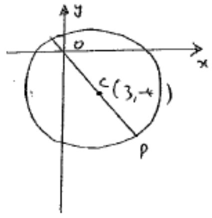
Question 14 (AEA 2005 Q1)

A point P lies on the curve with equation

$$x^2 + y^2 - 6x + 8y = 24.$$

Find the greatest and least possible values of the length OP, where O is the origin.

(6)

$(x-3)^2 + (y+4)^2 = 24+9+16 = 49$	M1
Curve is circle, centre $(3, -4)$, radius 7	A1
 $OC = \sqrt{3^2+4^2} = 5$	M1
Greatest length $OP = 5 + r$ (or least)	M1
$= \underline{\underline{12}}$	A1
Least length $= r - 5 = \underline{\underline{2}}$	A1
	(6)

1.7 ALGEBRAIC METHODS (ALG FRACTIONS, DIVIDING POLYS, FACTOR THEOREM, PROOF)

Question 1 (STEP I 2014 Q1)

All numbers referred to in this question are non-negative integers.

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

Solutions: (vi) 6

Question 2 (STEP I 2010 Q8)

(i) Suppose that a, b and c are integers that satisfy the equation

$$a^3 + 3b^3 = 9c^3.$$

Explain why a must be divisible by 3, and show further that both b and c must also be divisible by 3. Hence show that the only integer solution is $a = b = c = 0$.

(ii) Suppose that p, q and r are integers that satisfy the equation

$$p^4 + 2q^4 = 5r^4.$$

By considering the possible final digit of each term, or otherwise, show that p and q are divisible by 5. Hence show that the only integer solution is $p = q = r = 0$.

Question 3 (STEP I 2009 Q1)

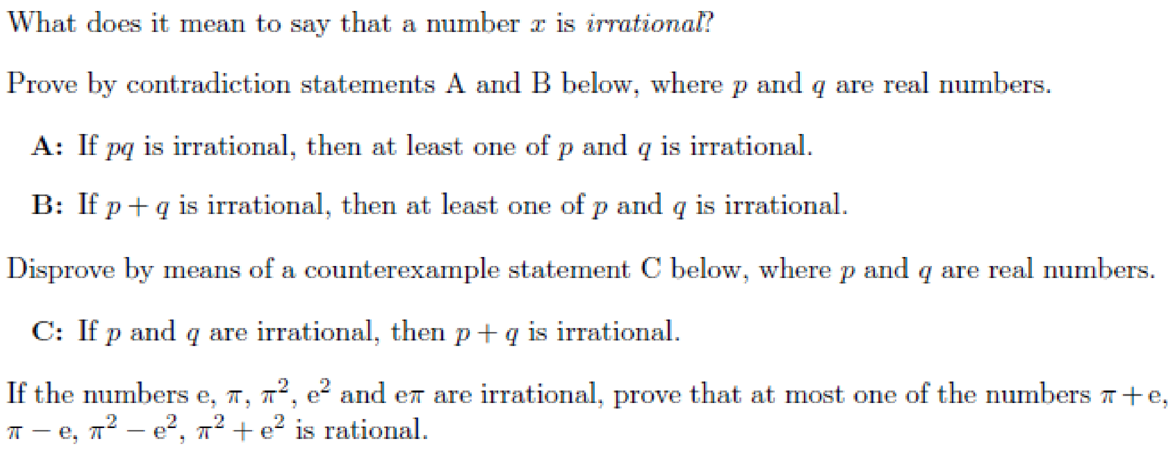
A *proper factor* of an integer N is a positive integer, not 1 or N, that divides N.

(i) Show that $3^2 \times 5^3$ has exactly 10 proper factors. Determine how many other integers of the form $3^m \times 5^n$ (where m and n are integers) have exactly 10 proper factors.

(ii) Let N be the smallest positive integer that has exactly 426 proper factors. Determine N, giving your answer in terms of its prime factors.

Solution: (ii) 15

Question 4 (STEP I 2008 Q1)



What does it mean to say that a number x is *irrational?*

Prove by contradiction statements A and B below, where p and q are real numbers.

A: If pq is irrational, then at least one of p and q is irrational.

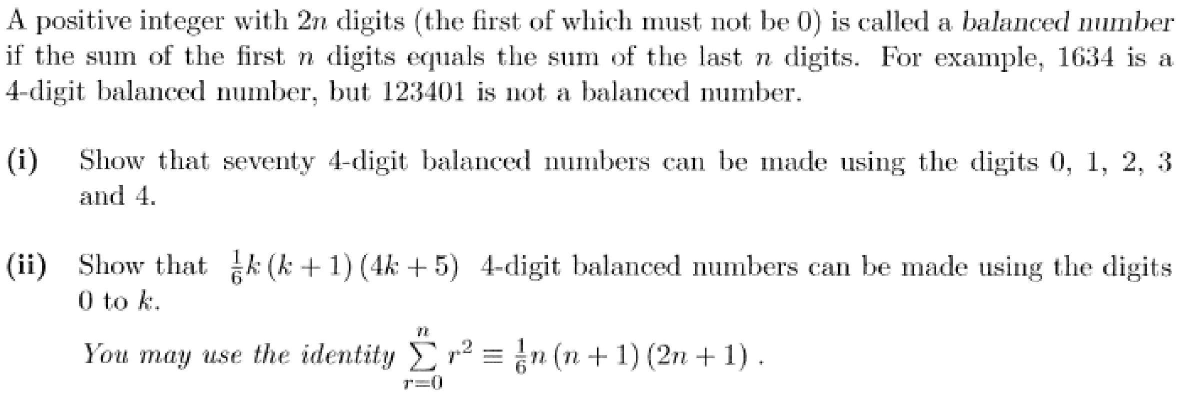
B: If $p+q$ is irrational, then at least one of p and q is irrational.

Disprove by means of a counterexample statement C below, where p and q are real numbers.

C: If p and q are irrational, then $p+q$ is irrational.

If the numbers e, π, π^2, e^2 and $\mathrm{e}\pi$ are irrational, prove that at most one of the numbers $\pi+\mathrm{e}$, $\pi-\mathrm{e}$, $\pi^2-\mathrm{e}^2$, $\pi^2+\mathrm{e}^2$ is rational.

Question 5 (STEP I 2007 Q1)



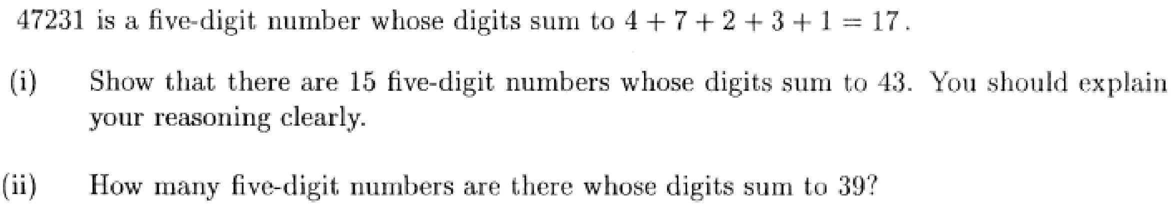
A positive integer with $2n$ digits (the first of which must not be 0) is called a *balanced number* if the sum of the first n digits equals the sum of the last n digits. For example, 1634 is a 4-digit balanced number, but 123401 is not a balanced number.

(i) Show that seventy 4-digit balanced numbers can be made using the digits 0, 1, 2, 3 and 4.

(ii) Show that $\frac{1}{6}k\,(k+1)\,(4k+5)$ 4-digit balanced numbers can be made using the digits 0 to k.

You may use the identity $\sum_{r=0}^{n} r^2 \equiv \frac{1}{6}n\,(n+1)\,(2n+1)$.

Question 6 (STEP I 2005 Q1)



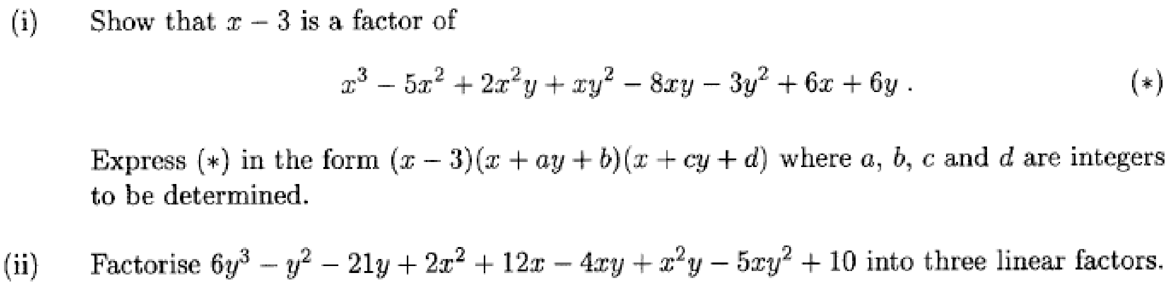
47231 is a five-digit number whose digits sum to $4+7+2+3+1=17$.

(i) Show that there are 15 five-digit numbers whose digits sum to 43. You should explain your reasoning clearly.

(ii) How many five-digit numbers are there whose digits sum to 39?

Solutions: (ii) 210 arrangements

Question 7 (STEP I 2004 Q3)



(i) Show that $x-3$ is a factor of

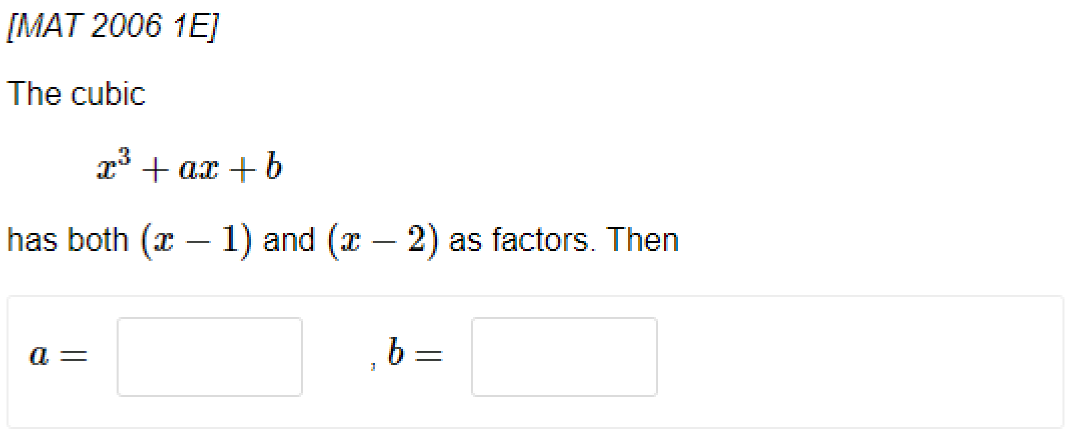
$$x^3-5x^2+2x^2y+xy^2-8xy-3y^2+6x+6y\,. \qquad (*)$$

Express $(*)$ in the form $(x-3)(x+ay+b)(x+cy+d)$ where a, b, c and d are integers to be determined.

(ii) Factorise $6y^3-y^2-21y+2x^2+12x-4xy+x^2y-5xy^2+10$ into three linear factors.

Solutions: (i) $(x-3)(x+y-2)(x+y)$ (ii) $(y+2)(x-2y+1)(x-3y+5)$

Question 8



[MAT 2006 1E]

The cubic

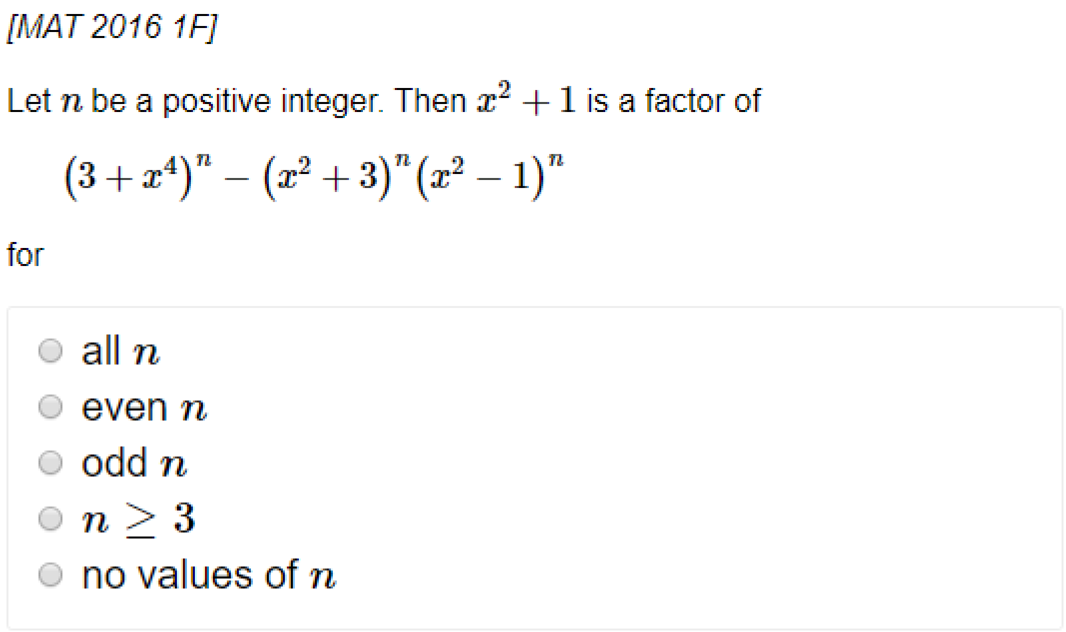
$$x^3 + ax + b$$

has both $(x-1)$ and $(x-2)$ as factors. Then

$a =$ ☐ , $b =$ ☐

Solution: $a = -7, b = 6$

Question 9



[MAT 2016 1F]

Let n be a positive integer. Then x^2+1 is a factor of

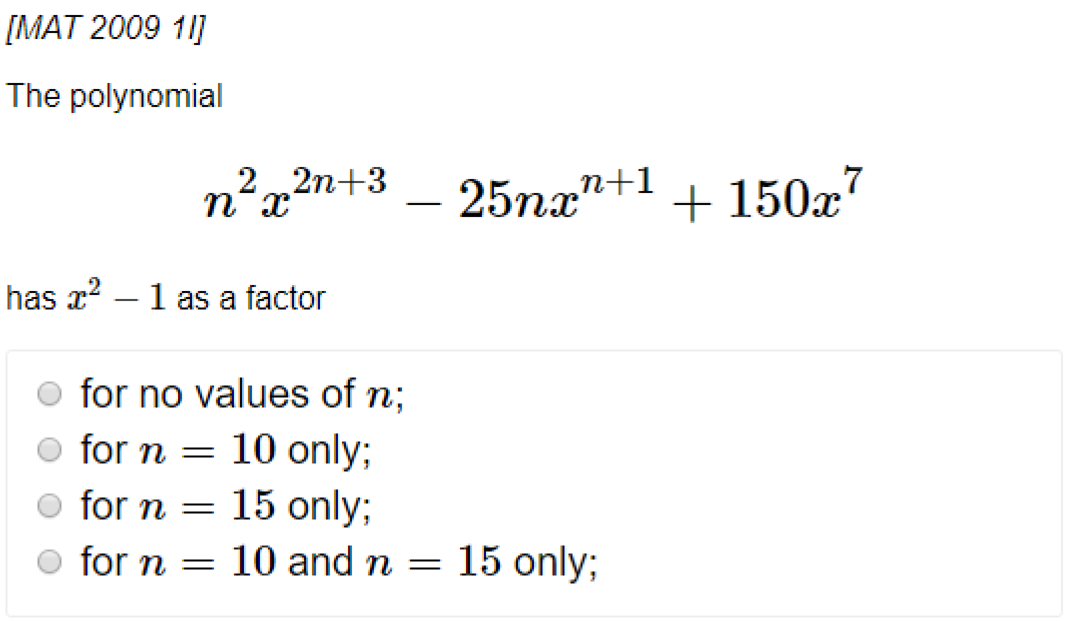
$$(3+x^4)^n - (x^2+3)^n(x^2-1)^n$$

for

- ○ all n
- ○ even n
- ○ odd n
- ○ $n \geq 3$
- ○ no values of n

Solution: Option 2

Question 10



[MAT 2009 1I]

The polynomial

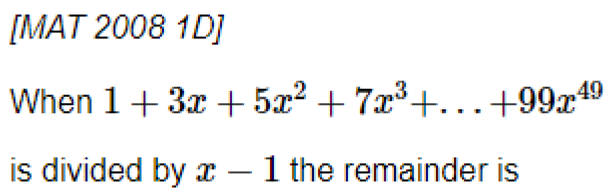
$$n^2x^{2n+3} - 25nx^{n+1} + 150x^7$$

has $x^2 - 1$ as a factor

- ○ for no values of n;
- ○ for $n = 10$ only;
- ○ for $n = 15$ only;
- ○ for $n = 10$ and $n = 15$ only;

Solution: Option 2

(Note: Knowledge of arithmetic series helpful)



[MAT 2008 1D]

When $1 + 3x + 5x^2 + 7x^3 + \ldots + 99x^{49}$

is divided by $x - 1$ the remainder is

Solution: 2500

1.8 BINOMIAL EXPANSION (INCLUDING FACTORIAL NOTATION)

Question 1 (STEP I 2013 Q6)

By considering the coefficient of x^r in the series for $(1+x)(1+x)^n$, or otherwise, obtain the following relation between binomial coefficients:

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \qquad (1 \leqslant r \leqslant n).$$

The sequence of numbers B_0, B_1, B_2, ... is defined by

$$B_{2m} = \sum_{j=0}^{m} \binom{2m-j}{j} \qquad \text{and} \qquad B_{2m+1} = \sum_{k=0}^{m} \binom{2m+1-k}{k}.$$

Show that $B_{n+2} - B_{n+1} = B_n$ $(n = 0, 1, 2, \ldots)$.

What is the relation between the sequence B_0, B_1, B_2, ... and the Fibonacci sequence F_0, F_1, F_2, ... defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 2$?

Solutions: (iii) $B_n = F_{n+1}$ for all n.

Question 2 (STEP I 2011 Q8)

(i) The numbers m and n satisfy

$$m^3 = n^3 + n^2 + 1\,. \qquad (*)$$

(a) Show that $m > n$. Show also that $m < n+1$ if and only if $2n^2 + 3n > 0$. Deduce that $n < m < n+1$ unless $-\frac{3}{2} \leqslant n \leqslant 0$.

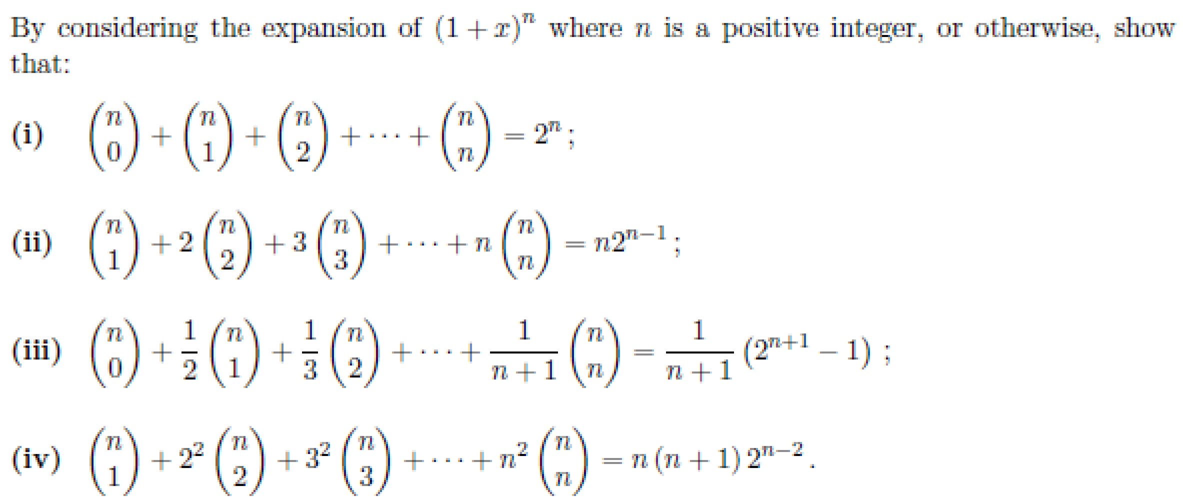
(b) Hence show that the only solutions of $(*)$ for which both m and n are integers are $(m,n) = (1,0)$ and $(m,n) = (1,-1)$.

(ii) Find all integer solutions of the equation

$$p^3 = q^3 + 2q^2 - 1\,.$$

Solutions: (ii) $(p,q) = (-1,0), (-1,-2), (0,-1)$

Question 3 (STEP I 2010 Q5)



By considering the expansion of $(1+x)^n$ where n is a positive integer, or otherwise, show that:

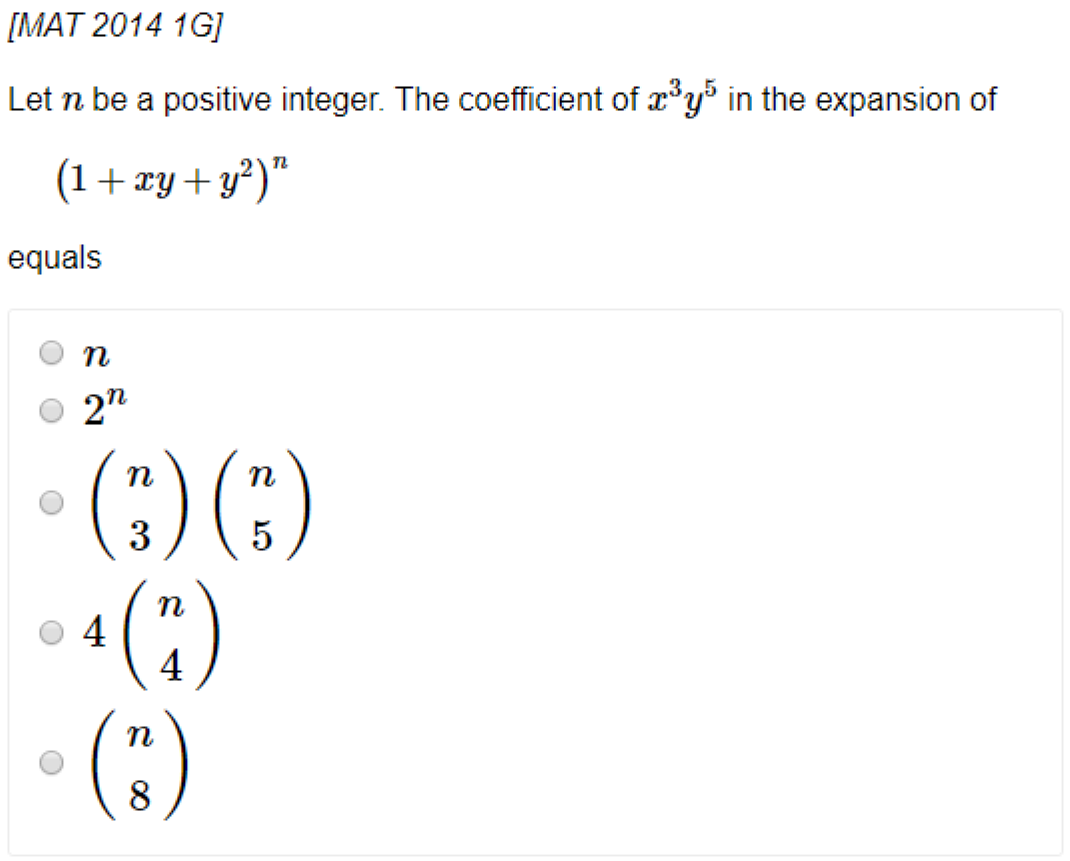
(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n\,;$

(ii) $\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}+\cdots+n\binom{n}{n}=n2^{n-1}\,;$

(iii) $\binom{n}{0}+\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}+\cdots+\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}\left(2^{n+1}-1\right)\,;$

(iv) $\binom{n}{1}+2^2\binom{n}{2}+3^2\binom{n}{3}+\cdots+n^2\binom{n}{n}=n\left(n+1\right)2^{n-2}\,.$

Question 4



[MAT 2014 1G]

Let n be a positive integer. The coefficient of x^3y^5 in the expansion of

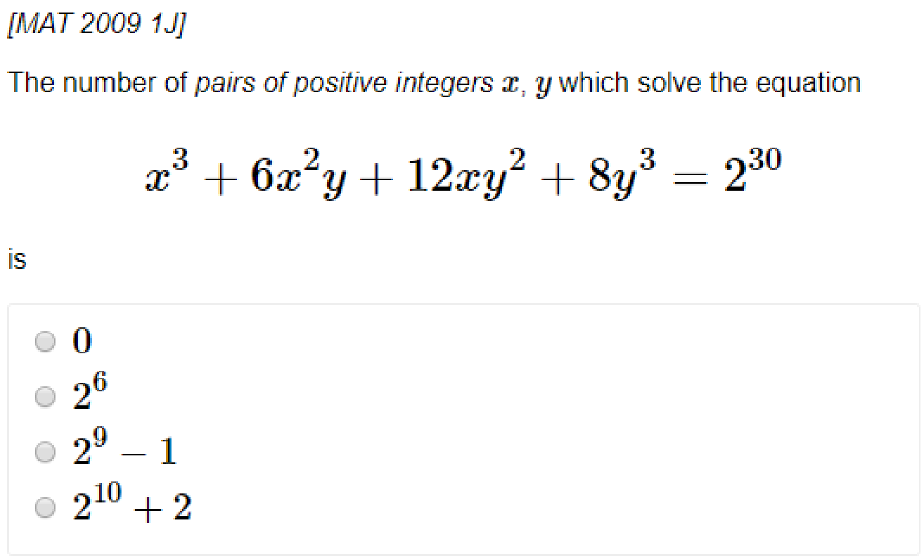
$$(1+xy+y^2)^n$$

equals

- ○ n
- ○ 2^n
- ○ $\binom{n}{3}\binom{n}{5}$
- ○ $4\binom{n}{4}$
- ○ $\binom{n}{8}$

Solution: Option 3

Question 5



[MAT 2009 1J]

The number of *pairs of positive integers* x, y which solve the equation

$$x^3+6x^2y+12xy^2+8y^3=2^{30}$$

is

- ○ 0
- ○ 2^6
- ○ 2^9-1
- ○ $2^{10}+2$

Solution: Option 3

Question 6 (AEA 2013 Q1)

In the binomial expansion of

$$\left(1+\frac{12n}{5}x\right)^n$$

the coefficients of x^2 and x^3 are equal and non-zero.

(a) Find the possible values of n.

(4)

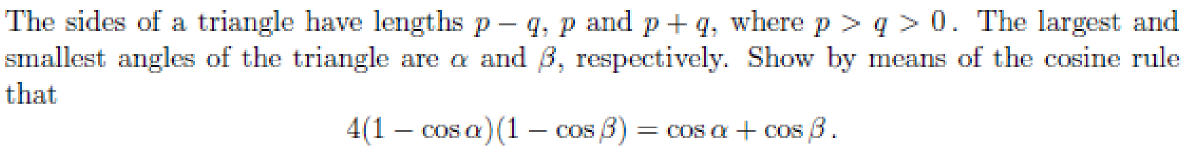
(b) State, giving a reason, which value of n gives a valid expansion when $x=\frac{1}{2}$

(2)

Question	Scheme	Marks	Notes
(a)	$\frac{n(n-1)}{2!}\left(\frac{12n}{5}\right)^2=\frac{n(n-1)(n-2)}{3!}\left(\frac{12n}{5}\right)^3$	M1	For attempting suitable equation. Ignore *x*s but must use binomial.
	$3\times5=n(n-2)\times12$ or $4n^2-8n-5=0$ (o.e.)	A1	Correct 3TQ in n May be other factors
	$(2n+1)(2n-5)=0$	dM1	Dep on 1st M1
	$n=-\frac{1}{2},\frac{5}{2}$	A1	Both & no others unless revoked later
		(4)	
(b)	$n=-\frac{1}{2}$ in $\left\lvert\frac{12nx}{5}\right\rvert<1$ gives $\lvert x\rvert<\frac{5}{6}$ and $n=\frac{5}{2}$ in $\left\lvert\frac{12nx}{5}\right\rvert$ gives $\lvert x\rvert<\frac{1}{6}$	M1	Attempt both cases Just check $n=-\frac{1}{2}$ SC B1
	So should choose $n=-\frac{1}{2}$	A1 (2)	
	May sub $x=\frac{1}{2}$ and get $\lvert n\rvert<\frac{5}{6}$ for M1 and A1 for stating $n=-\frac{1}{2}$	**(6)**	

1.9 TRIGONOMETRIC RATIOS (SINE/COSINE RULE, AREAS, GRAPHS)

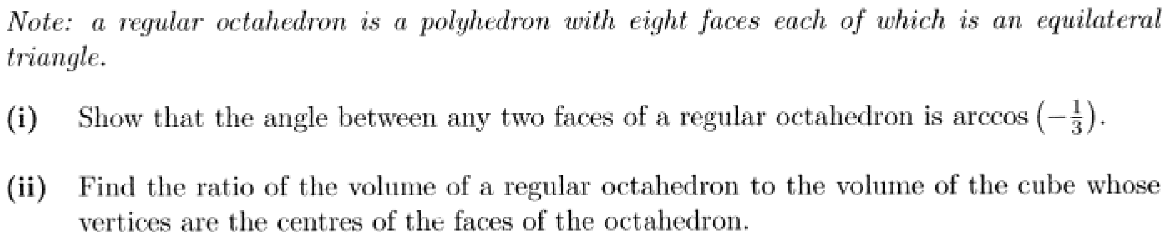
Question 1 (STEP I 2009 Q4i)



The sides of a triangle have lengths $p-q$, p and $p+q$, where $p>q>0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1-\cos\alpha)(1-\cos\beta)=\cos\alpha+\cos\beta\,.$$

Question 2 (STEP I 2007 Q5)



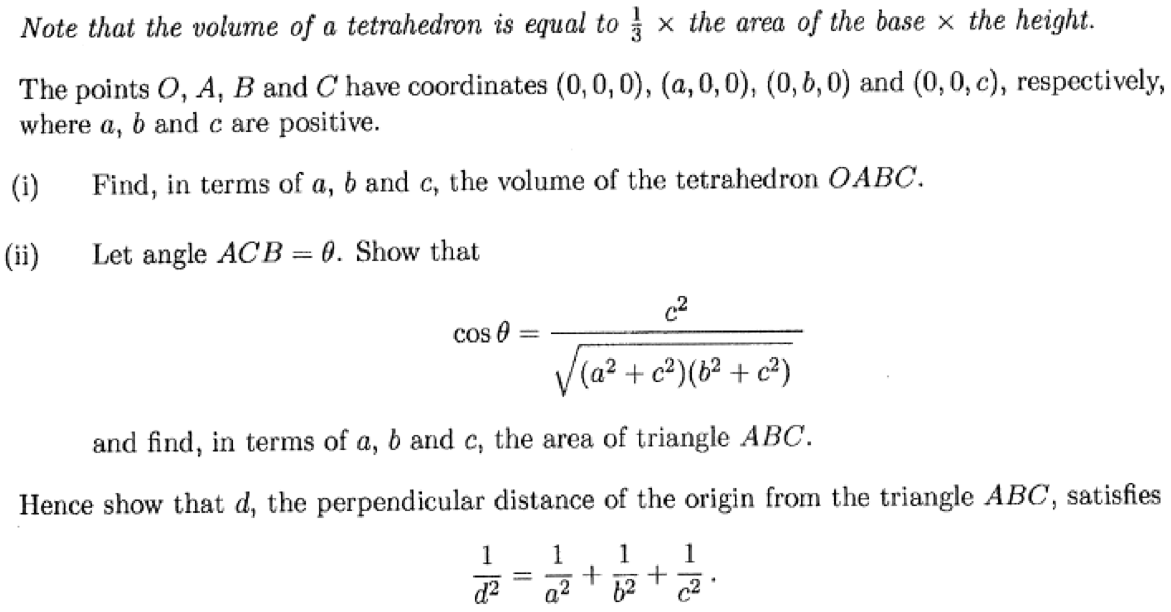
Note: a regular octahedron is a polyhedron with eight faces each of which is an equilateral triangle.

(i) Show that the angle between any two faces of a regular octahedron is $\arccos\left(-\frac{1}{3}\right)$.

(ii) Find the ratio of the volume of a regular octahedron to the volume of the cube whose vertices are the centres of the faces of the octahedron.

Solution: (ii) 9: 2

Question 3 (STEP I 2006 Q8)



Note that the volume of a tetrahedron is equal to $\frac{1}{3}$ $\times$ *the area of the base* $\times$ *the height.*

The points O, A, B and C have coordinates $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$, respectively, where a, b and c are positive.

(i) Find, in terms of a, b and c, the volume of the tetrahedron $OABC$.

(ii) Let angle $ACB=\theta$. Show that

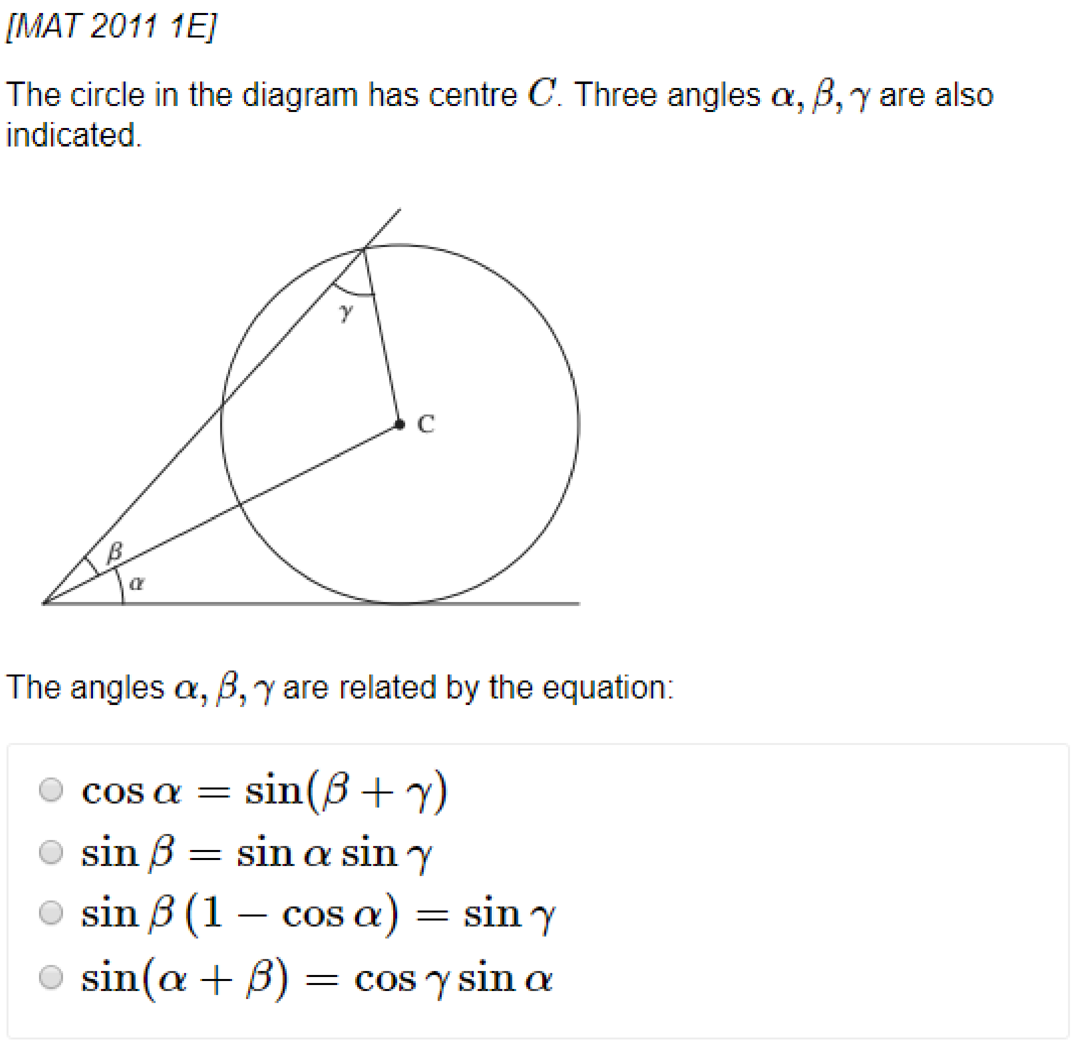
$$\cos\theta=\frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

and find, in terms of a, b and c, the area of triangle ABC.

Hence show that d, the perpendicular distance of the origin from the triangle ABC, satisfies

$$\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\,.$$

Question 4



[MAT 2011 1E]

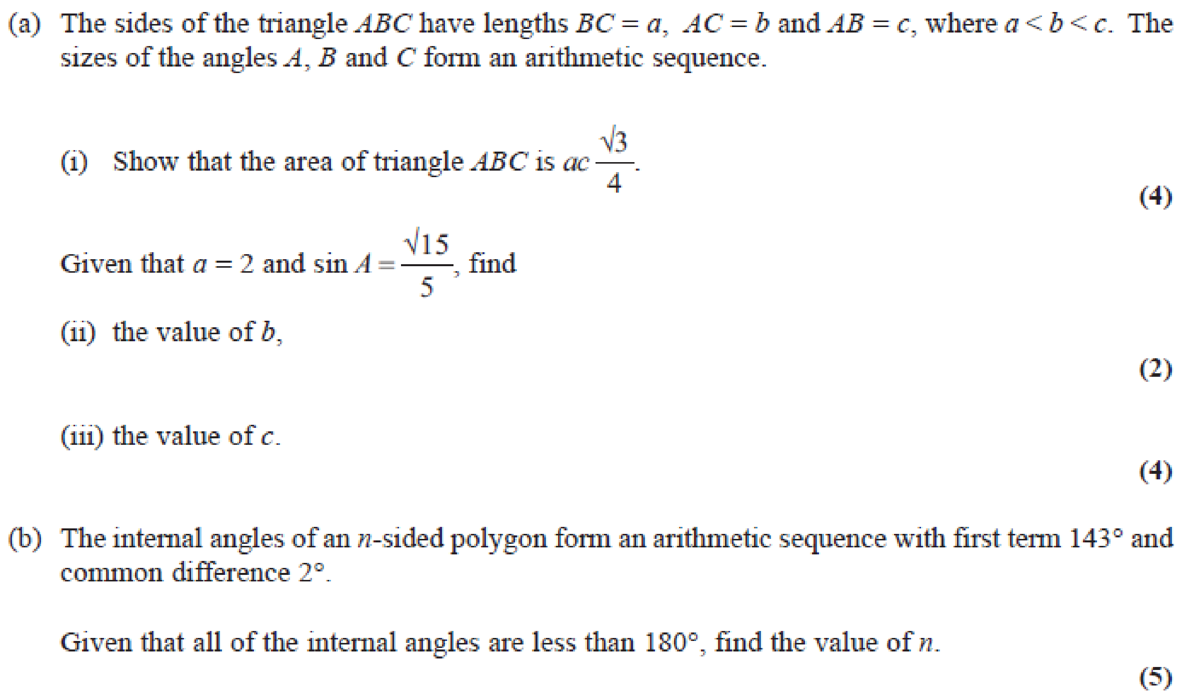
The circle in the diagram has centre C. Three angles α, β, γ are also indicated.

The angles α, β, γ are related by the equation:

○ $\cos\alpha = \sin(\beta+\gamma)$
○ $\sin\beta = \sin\alpha\sin\gamma$
○ $\sin\beta\,(1-\cos\alpha) = \sin\gamma$
○ $\sin(\alpha+\beta) = \cos\gamma\sin\alpha$

Solution: Option 2

Question 5 (AEA 2009 Q5)



(a) The sides of the triangle ABC have lengths $BC = a$, $AC = b$ and $AB = c$, where $a < b < c$. The sizes of the angles A, B and C form an arithmetic sequence.

(i) Show that the area of triangle ABC is $ac\dfrac{\sqrt{3}}{4}$.

(4)

Given that $a = 2$ and $\sin A = \dfrac{\sqrt{15}}{5}$, find

(ii) the value of b,

(2)

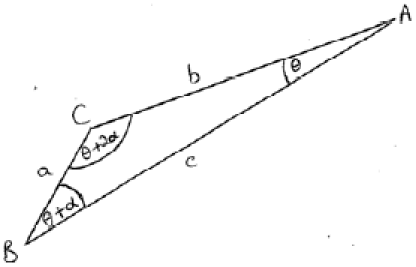
(iii) the value of c.

(4)

(b) The internal angles of an n-sided polygon form an arithmetic sequence with first term 143° and common difference 2°.

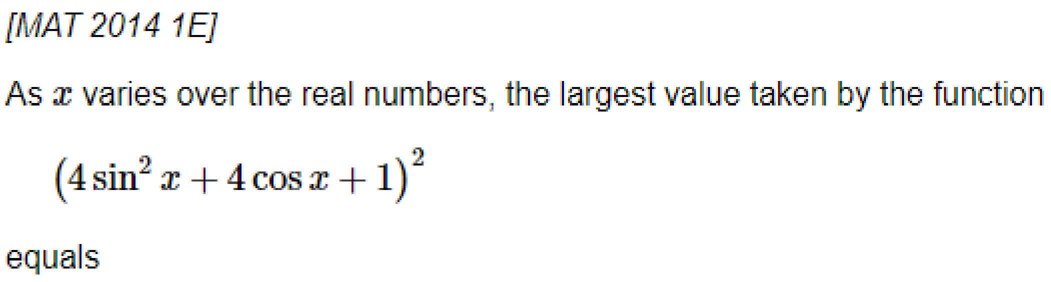
Given that all of the internal angles are less than 180°, find the value of n.

(5)

(a) (i)	 $\theta+(\theta+\alpha)+(\theta+2\alpha)=180$	M1		Equate S_3=180
	$3\theta+3\alpha \quad = 180$			
	$\therefore \hat{B}=(\theta+\alpha) \quad = 60°$	A1		Show $\hat{B}$ = 60°
	Area = $\frac{1}{2}ac\sin(\theta+\alpha)$	M1		Use of $\frac{1}{2}ac\sin B$
	$= \frac{1}{2}ac\frac{\sqrt{3}}{2} \qquad = \underline{\frac{ac\sqrt{3}}{4}} \quad (*)$	A1	(4)	
(ii)	Sine Rule $\frac{b}{\sin(\theta+\alpha)} = \frac{a}{\sin A}$ OR $\frac{1}{2}\ bc\sin A = \frac{ac\sqrt{3}}{4}$	M1		Correct use of sine rule or $\frac{1}{2}bc\sin A$ and (a)
	$\therefore \quad b \quad = 2\times\frac{5}{\sqrt{15}}\times\frac{\sqrt{3}}{2}=\underline{\sqrt{5}}$	A1	(2)	
(iii)	Cosine Rule $b^2 \quad = a^2+c^2-2ac\cos(\theta+\alpha)$	M1		Use of cos rule where all terms are known, except c.
	$5 \quad = 4+c^2-2\times2\times c\ \frac{1}{2}$			
	$0 \quad = c^2 - 2c - 1$ OR $c^2 - 2\sqrt{2}+1=0$	M1		Sub & simplify -> 3TQ
	$c \quad = \frac{2\pm\sqrt{4+4}}{2}$	M1		Solving
	$c \quad = \underline{1+\sqrt{2}}$ OR $(3+2\sqrt{2})^{1/2}$	A1	(4)	
(b)	$S_n = \frac{n}{2}[2\times143+2(n-1)] \quad = \{n(142+n)\}$	M1		For use of S_n needn't be simplified.
	Sum of internal angles $= 180(n-2)$	B1		
	$n(142+n) = 180(n-2) \Rightarrow \quad 0 = n^2-38n+360$	A1		Correct 3TQ.
	$0 = (n-19)^2-19^2+360$	M1		Attempt to solve relevant 3TQ
	$n-19 = \pm1 \quad (n = 20 \text{ or } 18)$			
	Internal angles all <180 $\quad u\ 20 = 143+19\times2 > 180$			] S+
	$u\ 18 = 143+17\times2 < 180$			
	$\therefore n = \underline{18}$	A1	(5) **[15]**	

1.10 TRIGONOMETRIC IDENTITIES & EQUATIONS

Question 1



[MAT 2014 1E]

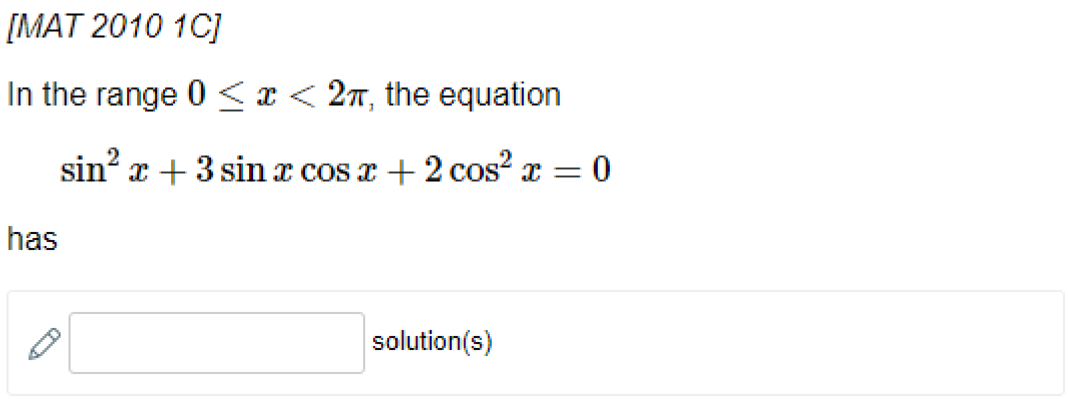
As x varies over the real numbers, the largest value taken by the function

$$\left(4\sin^2 x + 4\cos x + 1\right)^2$$

equals

Solution: 36

Question 2



[MAT 2010 1C]

In the range $0 \le x < 2\pi$, the equation

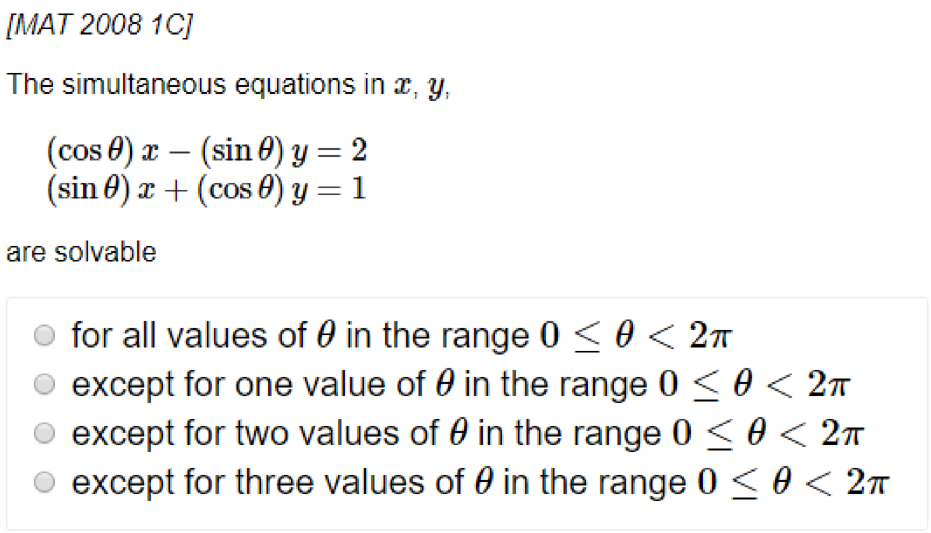
$$\sin^2 x + 3\sin x\cos x + 2\cos^2 x = 0$$

has

☐ solution(s)

Solution: 4 solutions

Question 3



[MAT 2008 1C]

The simultaneous equations in x, y,

$$(\cos\theta)\,x - (\sin\theta)\,y = 2$$
$$(\sin\theta)\,x + (\cos\theta)\,y = 1$$

are solvable

- ○ for all values of θ in the range $0 \le \theta < 2\pi$
- ○ except for one value of θ in the range $0 \le \theta < 2\pi$
- ○ except for two values of θ in the range $0 \le \theta < 2\pi$
- ○ except for three values of θ in the range $0 \le \theta < 2\pi$

Solution: Option 1

Question 1 (STEP I 2013 Q3)

For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X*Y$ and $Y*X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X*Y)*Z$ and $X*(Y*Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X*Y)*Z = (X*Z)*(Y*Z)$$

and obtain the corresponding result for $X*(Y*Z)$.

(iv) The points P_1, P_2, $\ldots$ are defined by $P_1 = X*Y$ and, for $n \geqslant 2$, $P_n = P_{n-1}*Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

Question 2 (STEP 2010 Q7)

Relative to a fixed origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. (The points O, A and B are not collinear.) The point C has position vector $\mathbf{c}$ given by

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b},$$

where α and β are positive constants with $\alpha + \beta < 1$. The lines OA and BC meet at the point P with position vector $\mathbf{p}$ and the lines OB and AC meet at the point Q with position vector $\mathbf{q}$. Show that

$$\mathbf{p} = \frac{\alpha\mathbf{a}}{1-\beta},$$

and write down $\mathbf{q}$ in terms of α, β and $\mathbf{b}$.

Show further that the point R with position vector $\mathbf{r}$ given by

$$\mathbf{r} = \frac{\alpha\mathbf{a} + \beta\mathbf{b}}{\alpha + \beta},$$

lies on the lines OC and AB.

The lines OB and PR intersect at the point S. Prove that $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.

Question 1 (STEP I 2014 Q8)

Let L_a denote the line joining the points $(a, 0)$ and $(0, 1-a)$, where $0 < a < 1$. The line L_b is defined similarly.

(i) Determine the point of intersection of L_a and L_b, where $a \neq b$.

(ii) Show that this point of intersection, in the limit as $b \to a$, lies on the curve C given by

$$y = (1 - \sqrt{x})^2 \qquad (0 < x < 1)\,.$$

(iii) Show that every tangent to C is of the form L_a for some a.

Solution: (i) $\big(ab, (1-a)(1-b)\big)$

Question 2 (STEP I 2012 Q2)

(i) Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

Question 3 (STEP I 2012 Q4)

The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}, \frac{1}{p+q}\right).$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

Question 4 (STEP I 2008 Q5)

The polynomial $\mathrm{p}(x)$ is given by

$$\mathrm{p}(x) = x^n + \sum_{r=0}^{n-1} a_r x^r \,,$$

where a_0, a_1, ... , a_{n-1} are fixed real numbers and $n \geqslant 1$. Let M be the greatest value of $|\mathrm{p}(x)|$ for $|x| \leqslant 1$. Then *Chebyshev's theorem* states that $M \geqslant 2^{1-n}$.

(i) Prove Chebyshev's theorem in the case $n = 1$ and verify that Chebyshev's theorem holds in the following cases:

(a) $\mathrm{p}(x) = x^2 - \frac{1}{2}$;

(b) $\mathrm{p}(x) = x^3 - x$.

(ii) Use Chebyshev's theorem to show that the curve $y = 64x^5 + 25x^4 - 66x^3 - 24x^2 + 3x + 1$ has at least one turning point in the interval $-1 \leqslant x \leqslant 1$.

Question 5 (STEP I 2007 Q8)

A curve is given by the equation

$$y = ax^3 - 6ax^2 + (12a + 12)\,x - (8a + 16) \,, \qquad (*)$$

where a is a real number. Show that this curve touches the curve with equation

$$y = x^3 \qquad (**)$$

at $(2\,,\,8)$. Determine the coordinates of any other point of intersection of the two curves.

(i) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 2$.

(ii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = 1$.

(iii) Sketch on the same axes the curves $(*)$ and $(**)$ when $a = -2$.

Solution: $\left(\frac{2a+4}{a-1}, \left[\frac{2a+4}{a-1}\right]^3\right)$ (i) Touch at (2,8), intersect at (8,512), no turning points (ii) touch at (2,8), do not intersect elsewhere, and has no turning points, (iii) Touch at (2,8), intersect at (0,0), turns at $x = 2 \pm \sqrt{2}$

Question 6 (STEP I 2006 Q2)

A small goat is tethered by a rope to a point at ground level on a side of a square barn which stands in a large horizontal field of grass. The sides of the barn are of length $2a$ and the rope is of length $4a$. Let A be the area of the grass that the goat can graze. Prove that $A \leqslant 14\pi a^2$ and determine the minimum value of A.

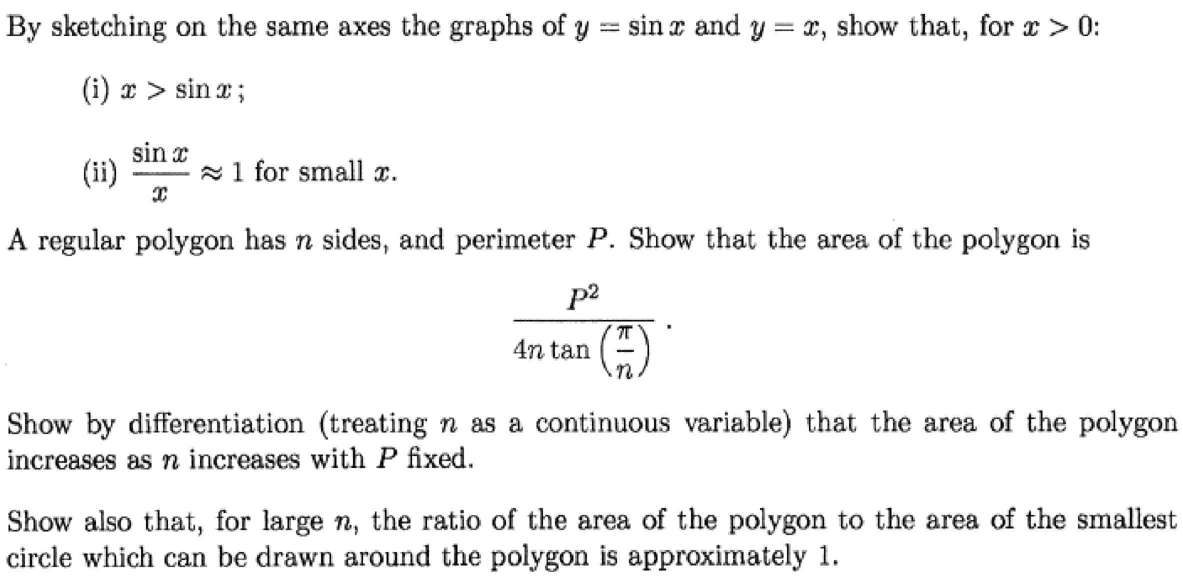
Solution:

Let the goat be tethered a distance x from a corner. Therefore, the goat can graze an area

$$A = \frac{16a^2\pi}{2} + \frac{(4a-x)^2\,\pi}{4} + \frac{(2a-x)^2\,\pi}{4} + \frac{(2a+x)^2\,\pi}{4} + \frac{(x)^2\,\pi}{4} = \frac{\pi}{4}\left(56a^2 + 4x^2 - 8ax\right)$$

So the area grazed $A = \pi\left[13a^2 + (x-a)^2\right]$. This is minimised when $x = a$, and maximised when $x = 0$ or $2a$ (since $0 \leqslant x \leqslant 2a$), hence $13\pi a^2 \leqslant A \leqslant 14\pi a^2$.

Question 7 (STEP I 2006 Q4)



By sketching on the same axes the graphs of $y = \sin x$ and $y = x$, show that, for $x > 0$:

(i) $x > \sin x$;

(ii) $\dfrac{\sin x}{x} \approx 1$ for small x.

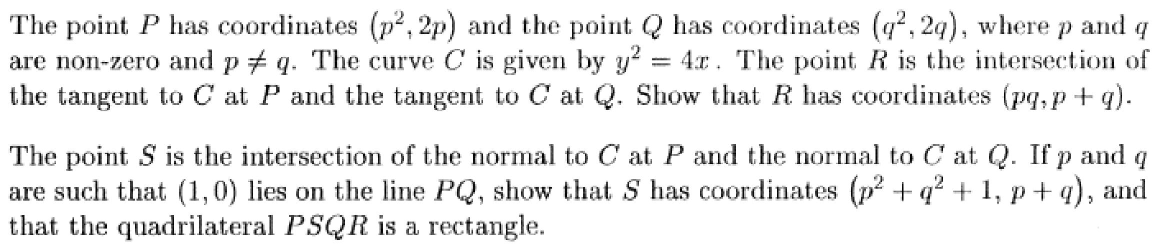
A regular polygon has n sides, and perimeter P. Show that the area of the polygon is

$$\frac{P^2}{4n\tan\left(\frac{\pi}{n}\right)}\,.$$

Show by differentiation (treating n as a continuous variable) that the area of the polygon increases as n increases with P fixed.

Show also that, for large n, the ratio of the area of the polygon to the area of the smallest circle which can be drawn around the polygon is approximately 1.

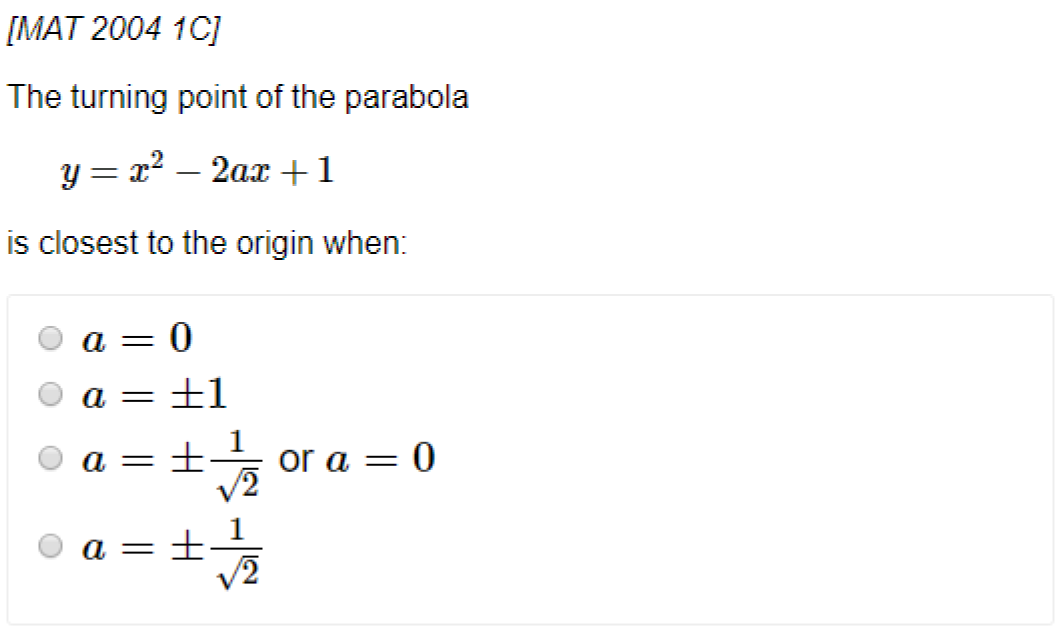
Question 8 (STEP I 2005 Q2)



The point P has coordinates $(p^2, 2p)$ and the point Q has coordinates $(q^2, 2q)$, where p and q are non-zero and $p \neq q$. The curve C is given by $y^2 = 4x$. The point R is the intersection of the tangent to C at P and the tangent to C at Q. Show that R has coordinates $(pq, p+q)$.

The point S is the intersection of the normal to C at P and the normal to C at Q. If p and q are such that $(1, 0)$ lies on the line PQ, show that S has coordinates $(p^2+q^2+1,\ p+q)$, and that the quadrilateral $PSQR$ is a rectangle.

Question 9



[MAT 2004 1C]

The turning point of the parabola

$$y = x^2 - 2ax + 1$$

is closest to the origin when:

○ $a = 0$

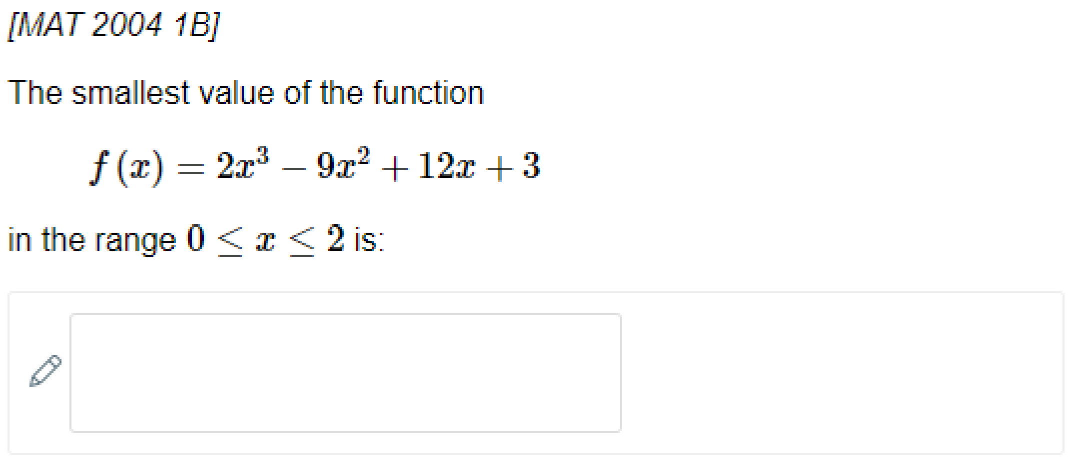
○ $a = \pm 1$

○ $a = \pm\frac{1}{\sqrt{2}}$ or $a = 0$

○ $a = \pm\frac{1}{\sqrt{2}}$

Solution: Option 4

Question 10



[MAT 2004 1B]

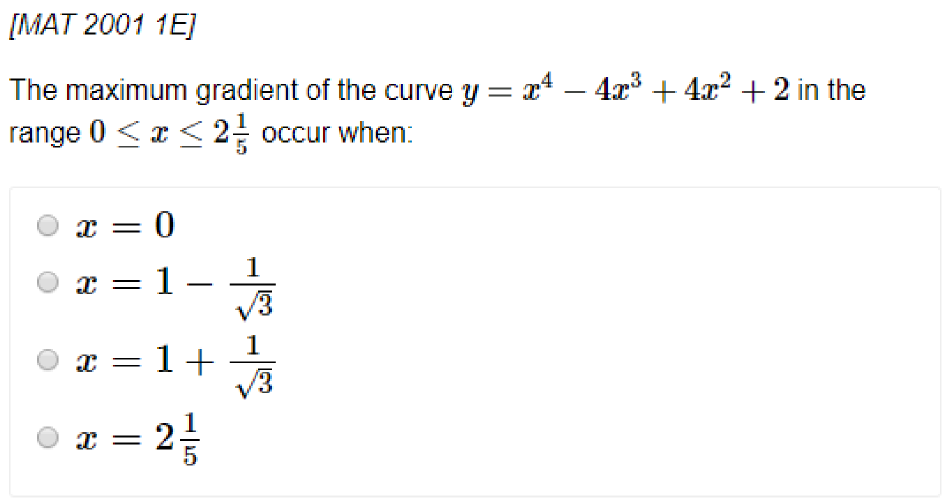
The smallest value of the function

$$f(x) = 2x^3 - 9x^2 + 12x + 3$$

in the range $0 \leq x \leq 2$ is:

Solution: 3

Question 11



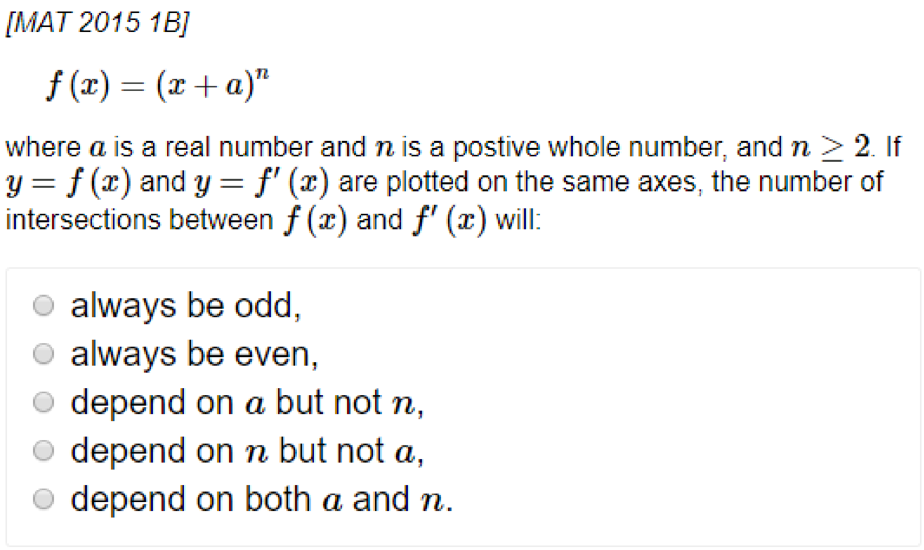
[MAT 2001 1E]

The maximum gradient of the curve $y = x^4 - 4x^3 + 4x^2 + 2$ in the range $0 \leq x \leq 2\frac{1}{5}$ occur when:

- ○ $x = 0$
- ○ $x = 1 - \frac{1}{\sqrt{3}}$
- ○ $x = 1 + \frac{1}{\sqrt{3}}$
- ○ $x = 2\frac{1}{5}$

Solution: Option 4

Question 12



[MAT 2015 1B]

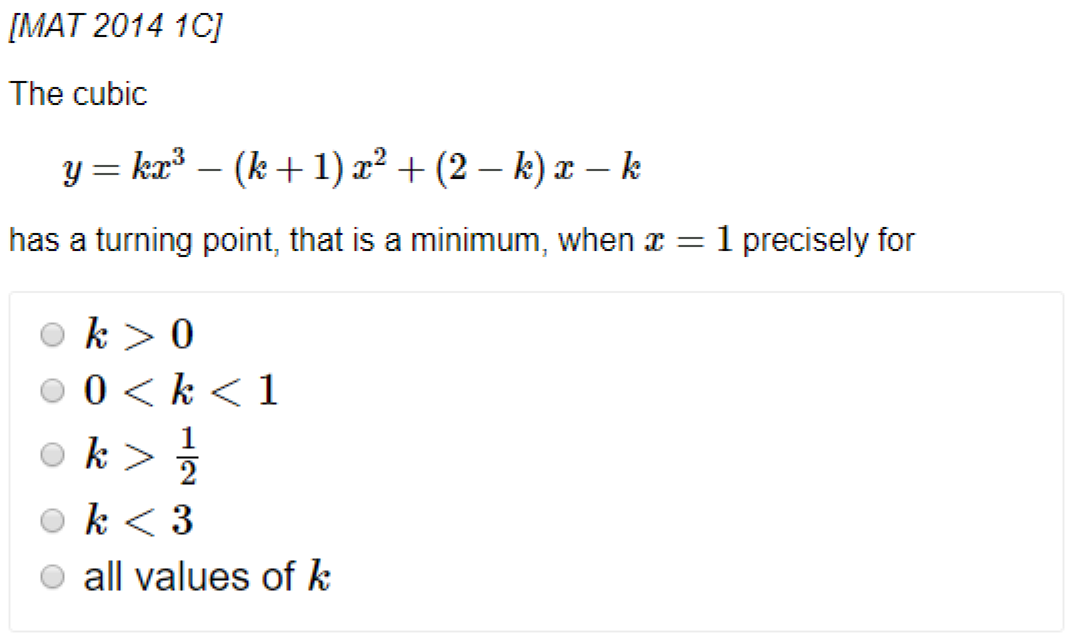
$$f(x) = (x + a)^n$$

where a is a real number and n is a postive whole number, and $n \geq 2$. If $y = f(x)$ and $y = f'(x)$ are plotted on the same axes, the number of intersections between $f(x)$ and $f'(x)$ will:

- ○ always be odd,
- ○ always be even,
- ○ depend on a but not n,
- ○ depend on n but not a,
- ○ depend on both a and n.

Solution: Option 2

Question 13



[MAT 2014 1C]

The cubic

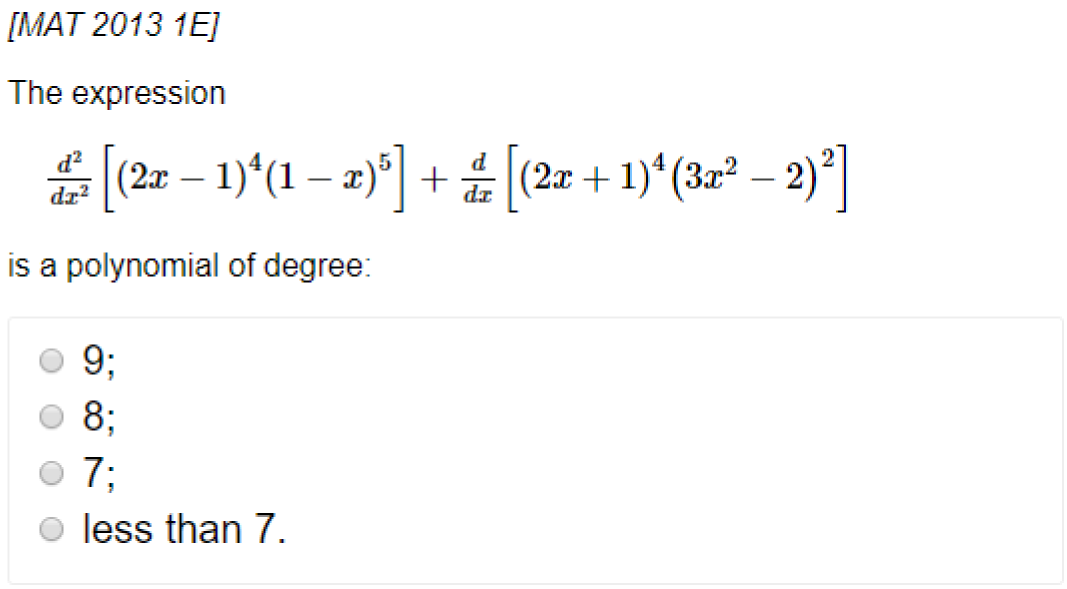
$$y = kx^3 - (k+1)\,x^2 + (2-k)\,x - k$$

has a turning point, that is a minimum, when $x = 1$ precisely for

○ $k > 0$
○ $0 < k < 1$
○ $k > \frac{1}{2}$
○ $k < 3$
○ all values of k

Solution: $k > \frac{1}{2}$

Question 14



[MAT 2013 1E]

The expression

$$\frac{d^2}{dx^2}\left[(2x-1)^4(1-x)^5\right] + \frac{d}{dx}\left[(2x+1)^4\left(3x^2-2\right)^2\right]$$

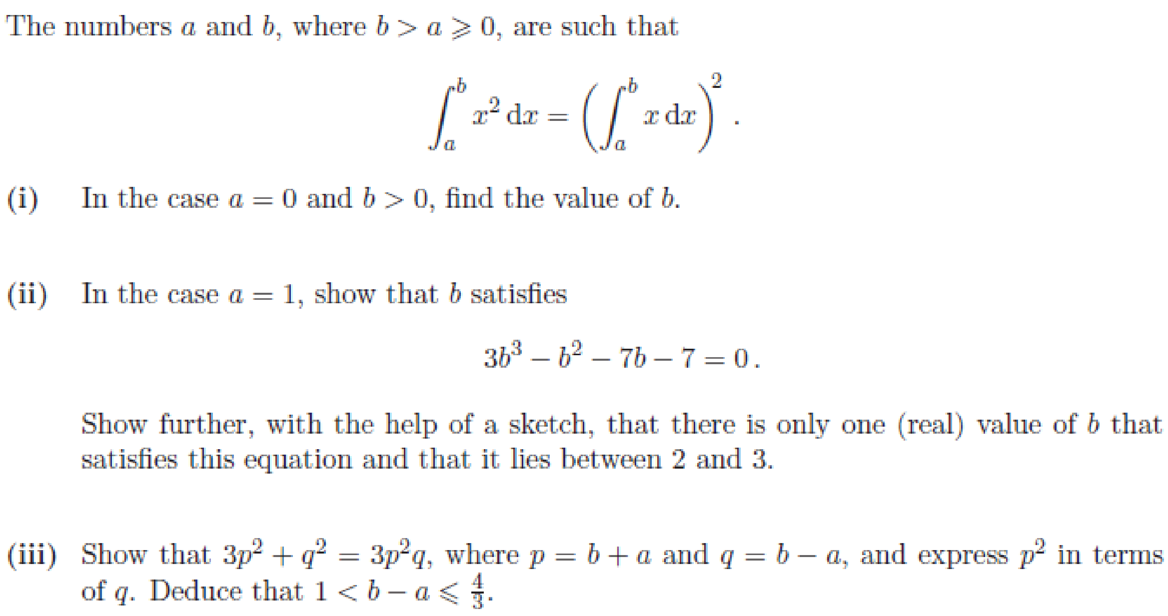
is a polynomial of degree:

○ 9;
○ 8;
○ 7;
○ less than 7.

Solution: 8

1.13 INTEGRATION (INTEGRATING POLYNOMIALS, AREAS UNDER/BETWEEN CURVES)

Question 1 (STEP I 2014 Q3)



The numbers a and b, where $b > a \geqslant 0$, are such that

$$\int_a^b x^2 \, \mathrm{d}x = \left(\int_a^b x \, \mathrm{d}x \right)^2 .$$

(i) In the case $a = 0$ and $b > 0$, find the value of b.

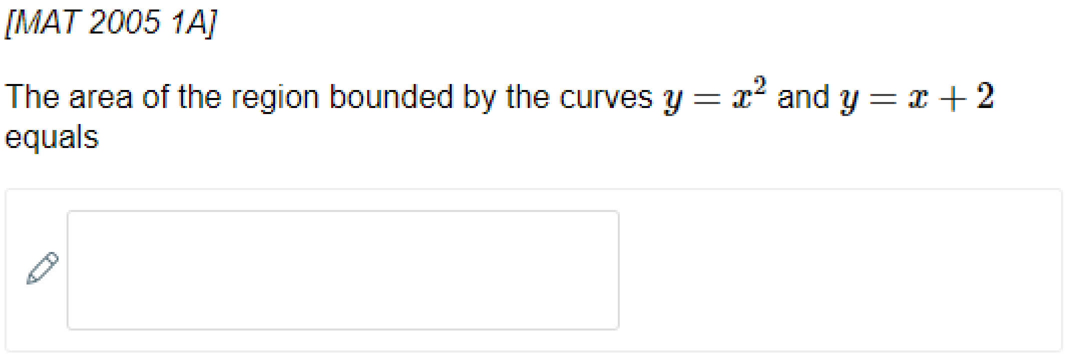
(ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0 .$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

(iii) Show that $3p^2 + q^2 = 3p^2 q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q. Deduce that $1 < b - a \leqslant \frac{4}{3}$.

Question 2

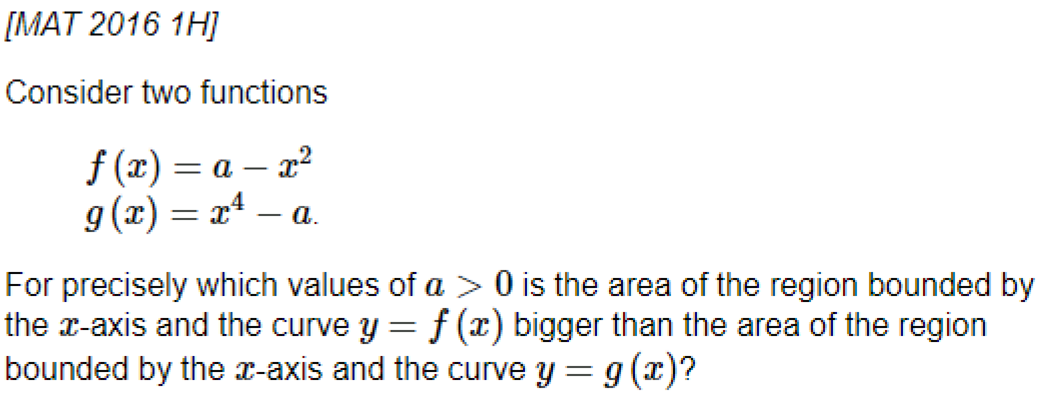


[MAT 2005 1A]

The area of the region bounded by the curves $y = x^2$ and $y = x + 2$ equals

Solution: $\frac{9}{2}$

Question 3



[MAT 2016 1H]

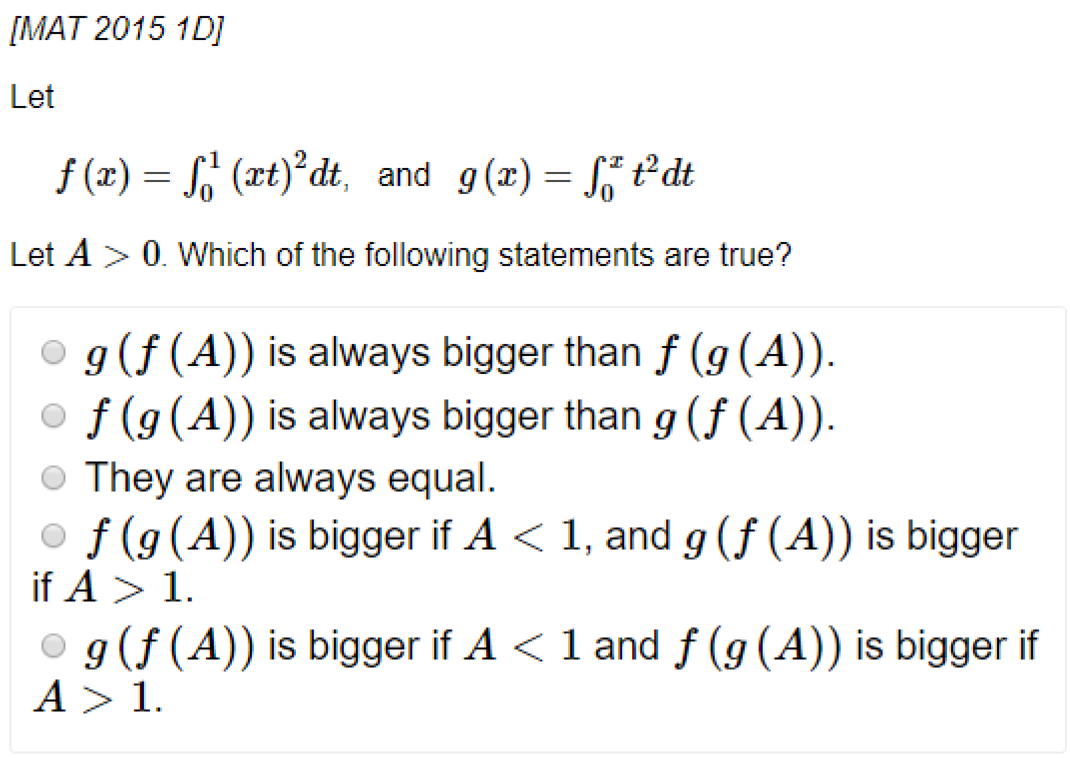
Consider two functions

$$f(x) = a - x^2$$
$$g(x) = x^4 - a.$$

For precisely which values of $a > 0$ is the area of the region bounded by the x-axis and the curve $y = f(x)$ bigger than the area of the region bounded by the x-axis and the curve $y = g(x)$?

Solution: $a > \left(\frac{6}{5}\right)^4$

Question 4



[MAT 2015 1D]

Let

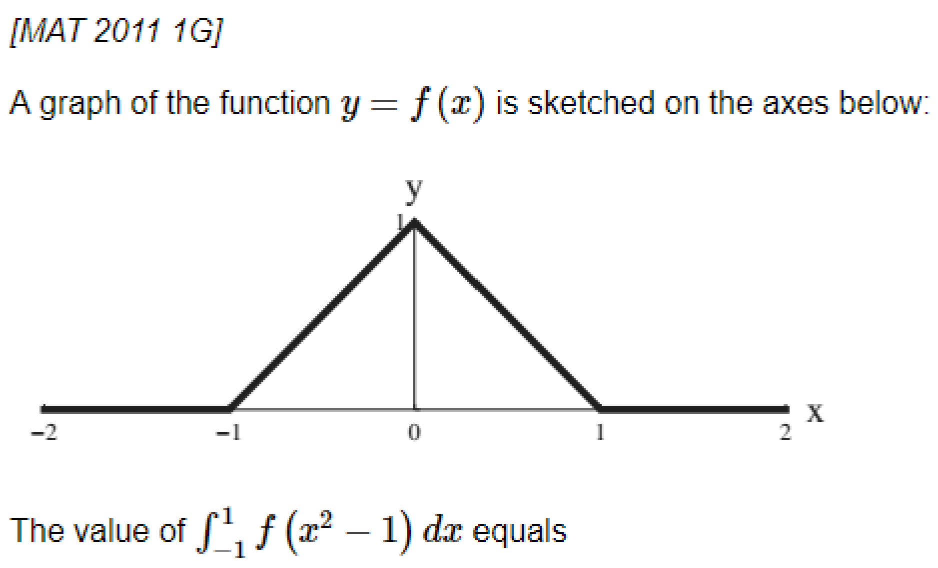
$$f(x) = \int_0^1 (xt)^2 dt, \quad \text{and} \quad g(x) = \int_0^x t^2 dt$$

Let $A > 0$. Which of the following statements are true?

- ○ $g(f(A))$ is always bigger than $f(g(A))$.
- ○ $f(g(A))$ is always bigger than $g(f(A))$.
- ○ They are always equal.
- ○ $f(g(A))$ is bigger if $A < 1$, and $g(f(A))$ is bigger if $A > 1$.
- ○ $g(f(A))$ is bigger if $A < 1$ and $f(g(A))$ is bigger if $A > 1$.

Solution: Option 2

Question 5



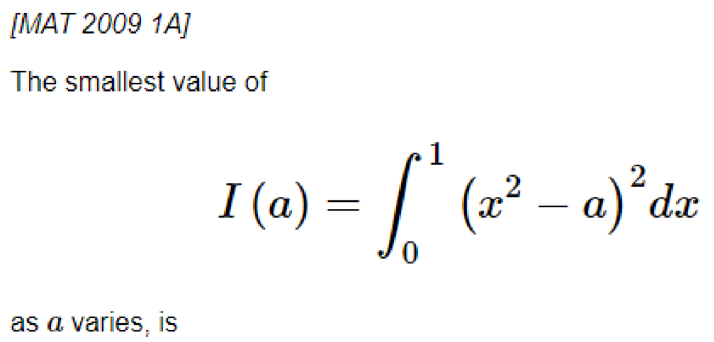
[MAT 2011 1G]

A graph of the function $y = f(x)$ is sketched on the axes below:

The value of $\int_{-1}^{1} f(x^2 - 1)\, dx$ equals

Solution: $\frac{2}{3}$

Question 6



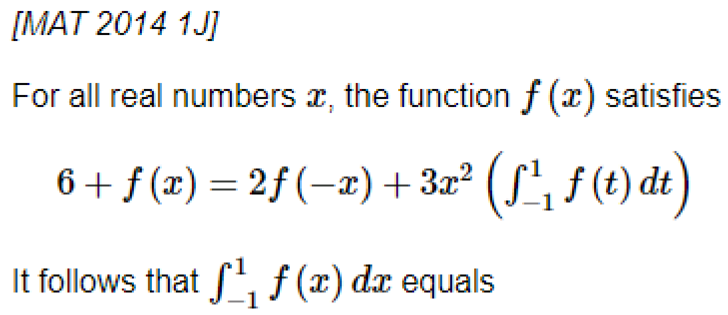
[MAT 2009 1A]

The smallest value of

$$I\left(a\right)=\int_0^1\left(x^2-a\right)^2dx$$

as a varies, is

Solution: $\frac{4}{45}$



[MAT 2014 1J]

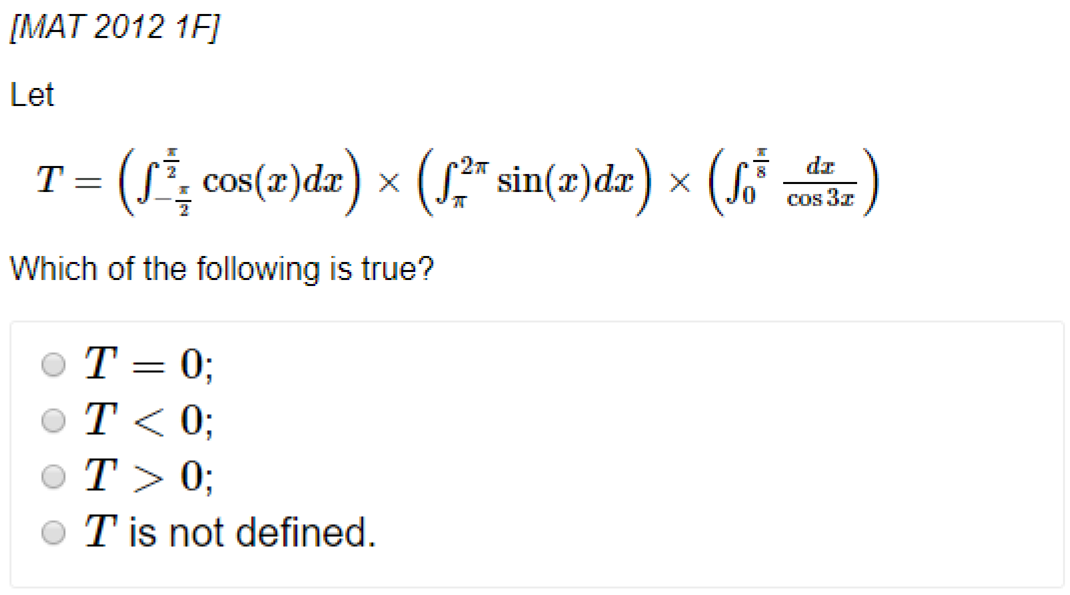
For all real numbers x, the function $f\left(x\right)$ satisfies

$$6+f\left(x\right)=2f\left(-x\right)+3x^2\left(\int_{-1}^{1}f\left(t\right)dt\right)$$

It follows that $\int_{-1}^{1}f\left(x\right)dx$ equals

Solution: 4

Question 7



[MAT 2012 1F]

Let

$$T=\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)dx\right)\times\left(\int_{\pi}^{2\pi}\sin(x)dx\right)\times\left(\int_0^{\frac{\pi}{8}}\frac{dx}{\cos 3x}\right)$$

Which of the following is true?

- ○ $T=0$;
- ○ $T<0$;
- ○ $T>0$;
- ○ T is not defined.

Solution: $T<0$

Question 8

[MAT 2010 1I]

For a positive number a, let

$$I(a) = \int_0^a \left(4 - 2^{x^2}\right) dx.$$

Then $\frac{dI}{da} = 0$ when:

Solution: $a = \sqrt{2}$

Question 9

[MAT 2007 1H]

Given a function $f(x)$, you are told that

$$\int_0^1 3f(x)\,dx + \int_1^2 2f(x)\,dx = 7$$
$$\int_0^2 f(x)\,dx + \int_1^2 f(x)\,dx = 1$$

It follows that $\int_0^2 f(x)\,dx$ equals:

Solution: 2

Question 10 (AEA 2007 Q2)

(*a*) On the same diagram, sketch $y = x$ and $y = \sqrt{x}$, for $x \geq 0$, and mark clearly the coordinates of the points of intersection of the two graphs.

(2)

(*b*) With reference to your sketch, explain why there exists a value a of x ($a > 1$) such that

$$\int_0^a x \ \mathrm{d}x = \int_0^a \sqrt{x} \ \mathrm{d}x.$$

(2)

(*c*) Find the exact value of a.

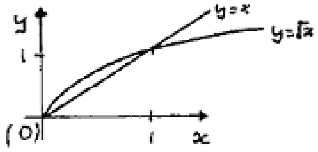
(4)

(*d*) Hence, or otherwise, find a non-constant function $\mathrm{f}(x)$ and a constant b ($b \neq 0$) such that

$$\int_{-b}^{b} \mathrm{f}(x) \ \mathrm{d}x = \int_{-b}^{b} \sqrt{[\mathrm{f}(x)]} \ \mathrm{d}x.$$

(2)

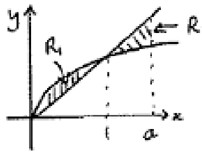
(a)



Relative shapes B1

0 or (0,0) implied and (1,1) On axes is OK. B1 (2)

(b) [$R_1 = \frac{1}{6}$]



As a increases from 1 R_2 increases. Choose a so that $R_2 = R_1$ then areas are the same.

Diagram with regions or mention of areas. B1g

Full argument B1h (2)

(c) $\int_0^a x\,dx = \int_0^a x^{\frac{1}{2}}dx \Rightarrow \left[\frac{x^2}{2}\right]_0^a = \left[\frac{2}{3}x^{3/2}\right]_0^a$

Attempt both integrals – one correct M1

$\Rightarrow \quad \frac{a^2}{2} = \frac{2}{3}a^{3/2}$

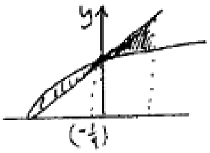
A correct equation in a A1

$\Rightarrow a^{3/2}(3a^{\frac{1}{2}} - 4) = 0 \quad \rightarrow a^{1/2} = 4/3$ o.e.

Attempt to solve $\rightarrow a^{\wedge} = k$ M1

$a = \underline{\underline{\frac{16}{9}}}$ A1 (4)

(d)



Translate $\frac{1}{2}a$ ←.

$f(x) = \underline{\underline{x + \frac{8}{9}}}$

$x + \frac{a}{2} = f(x)$ (Any suitable $f(x) + b$) B1 ∫

$\underline{\underline{b = \frac{8}{9}}}$

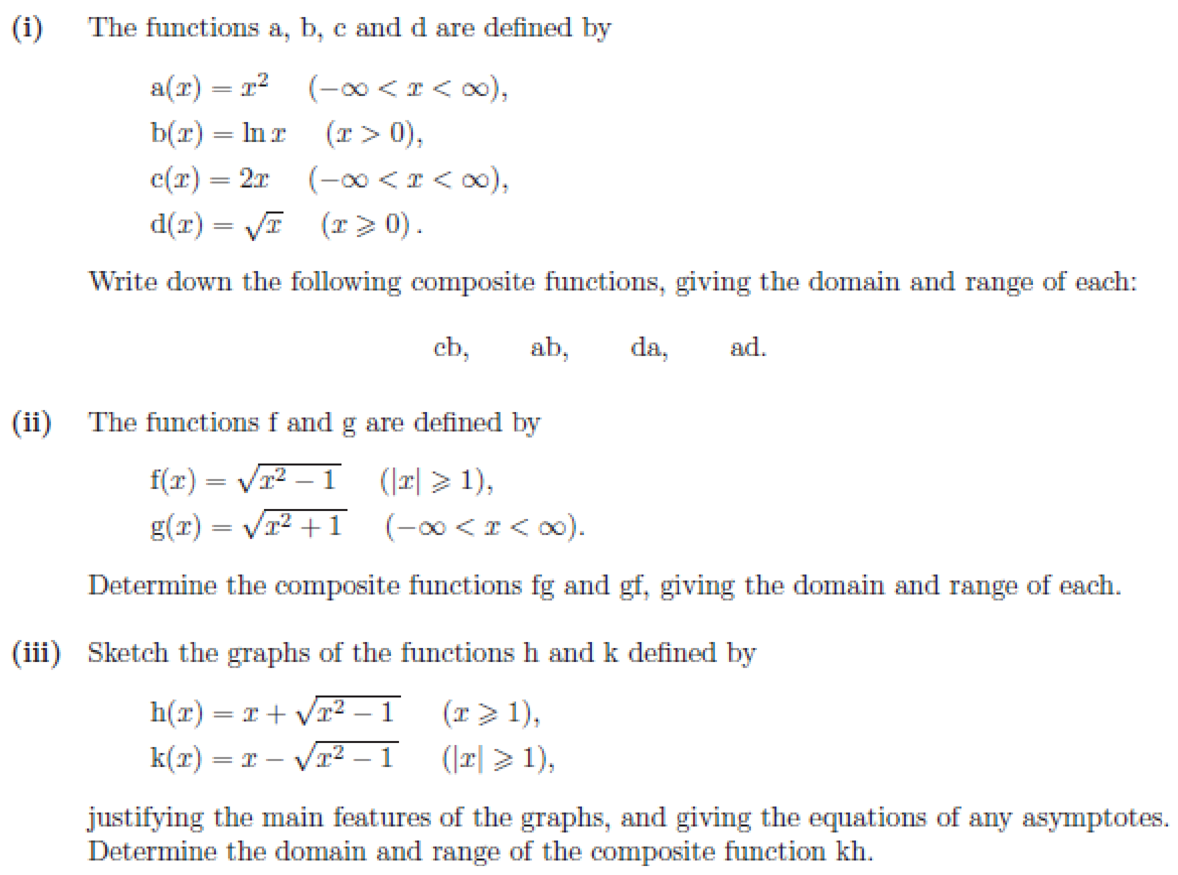
$\frac{a}{2} = b$ ∫ their a. B1 ∫ (2)

⑩

S.C. if $b = \beta$ and $f(x) = x + \beta$ score B1 only

1.14 EXPONENTIALS & LOGARITHMS (EXPONENTIAL MODELLING, SOLVING LOG EQUATIONS)

Question 1 (STEP I 2013 Q8)



(i) The functions a, b, c and d are defined by

$$\begin{aligned} \mathrm{a}(x) &= x^2 \quad (-\infty < x < \infty), \\ \mathrm{b}(x) &= \ln x \quad (x > 0), \\ \mathrm{c}(x) &= 2x \quad (-\infty < x < \infty), \\ \mathrm{d}(x) &= \sqrt{x} \quad (x \geqslant 0)\,. \end{aligned}$$

Write down the following composite functions, giving the domain and range of each:

cb, ab, da, ad.

(ii) The functions f and g are defined by

$$\begin{aligned} \mathrm{f}(x) &= \sqrt{x^2-1} \quad (|x| \geqslant 1), \\ \mathrm{g}(x) &= \sqrt{x^2+1} \quad (-\infty < x < \infty). \end{aligned}$$

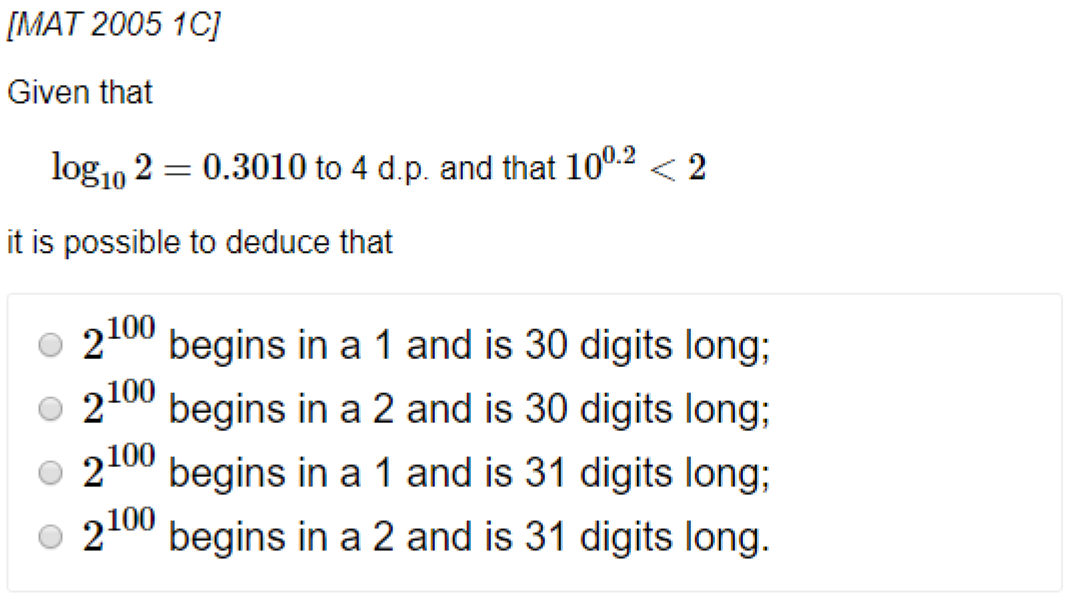
Determine the composite functions fg and gf, giving the domain and range of each.

(iii) Sketch the graphs of the functions h and k defined by

$$\begin{aligned} \mathrm{h}(x) &= x + \sqrt{x^2-1} \quad (x \geqslant 1), \\ \mathrm{k}(x) &= x - \sqrt{x^2-1} \quad (|x| \geqslant 1), \end{aligned}$$

justifying the main features of the graphs, and giving the equations of any asymptotes. Determine the domain and range of the composite function kh.

Question 2



[MAT 2005 1C]

Given that

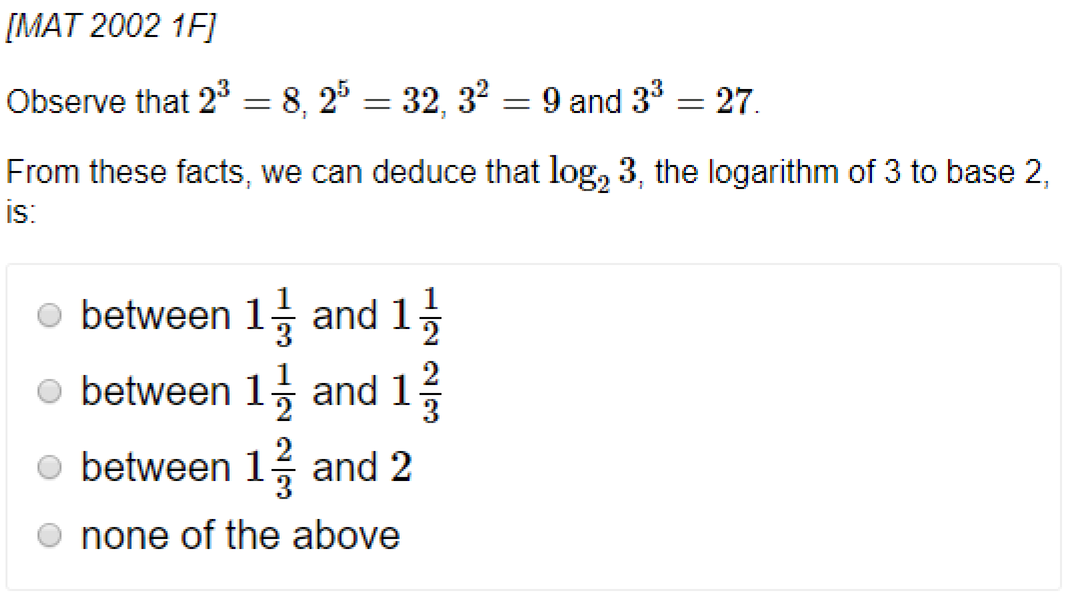
$\log_{10} 2 = 0.3010$ to 4 d.p. and that $10^{0.2} < 2$

it is possible to deduce that

○ 2^{100} begins in a 1 and is 30 digits long;
○ 2^{100} begins in a 2 and is 30 digits long;
○ 2^{100} begins in a 1 and is 31 digits long;
○ 2^{100} begins in a 2 and is 31 digits long.

Solution: Option 3

Question 3



[MAT 2002 1F]

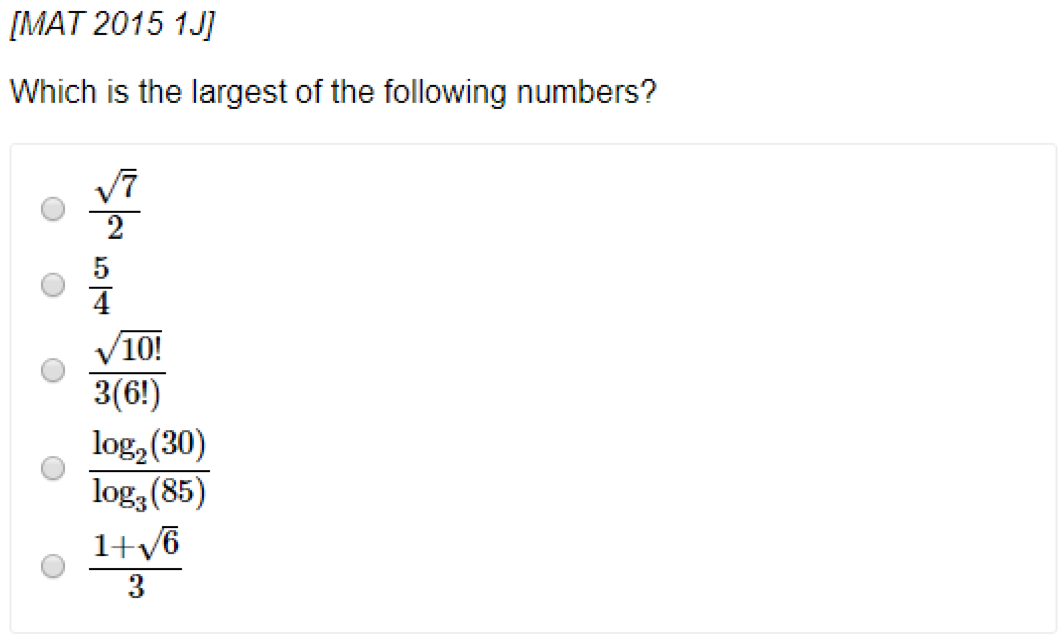
Observe that $2^3 = 8$, $2^5 = 32$, $3^2 = 9$ and $3^3 = 27$.

From these facts, we can deduce that $\log_2 3$, the logarithm of 3 to base 2, is:

- ○ between $1\frac{1}{3}$ and $1\frac{1}{2}$
- ○ between $1\frac{1}{2}$ and $1\frac{2}{3}$
- ○ between $1\frac{2}{3}$ and 2
- ○ none of the above

Solution: Option 2

Question 4



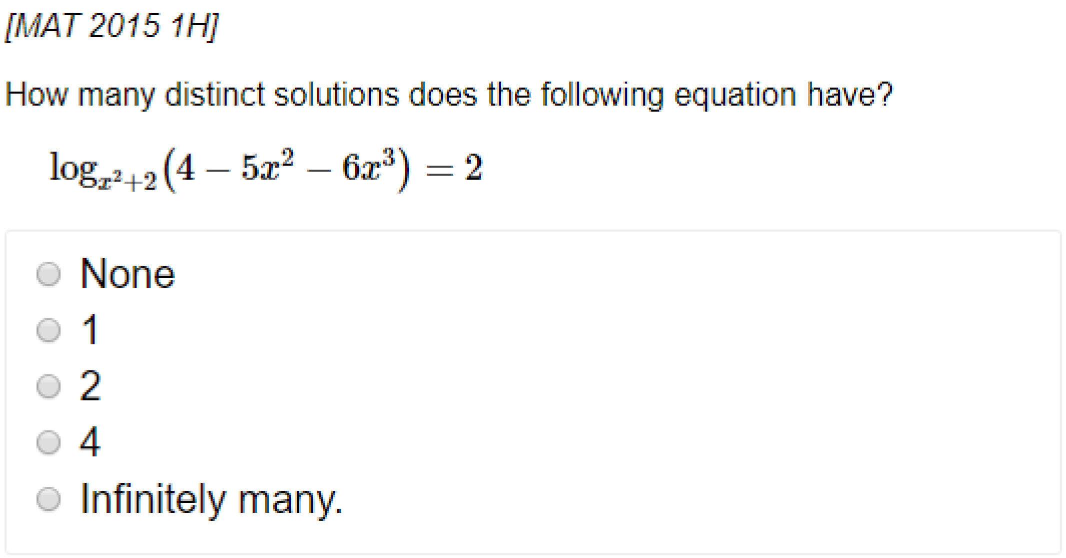
[MAT 2015 1J]

Which is the largest of the following numbers?

- ○ $\frac{\sqrt{7}}{2}$
- ○ $\frac{5}{4}$
- ○ $\frac{\sqrt{10!}}{3(6!)}$
- ○ $\frac{\log_2(30)}{\log_3(85)}$
- ○ $\frac{1+\sqrt{6}}{3}$

Solution: $\frac{\sqrt{7}}{2}$

Question 5



[MAT 2015 1H]

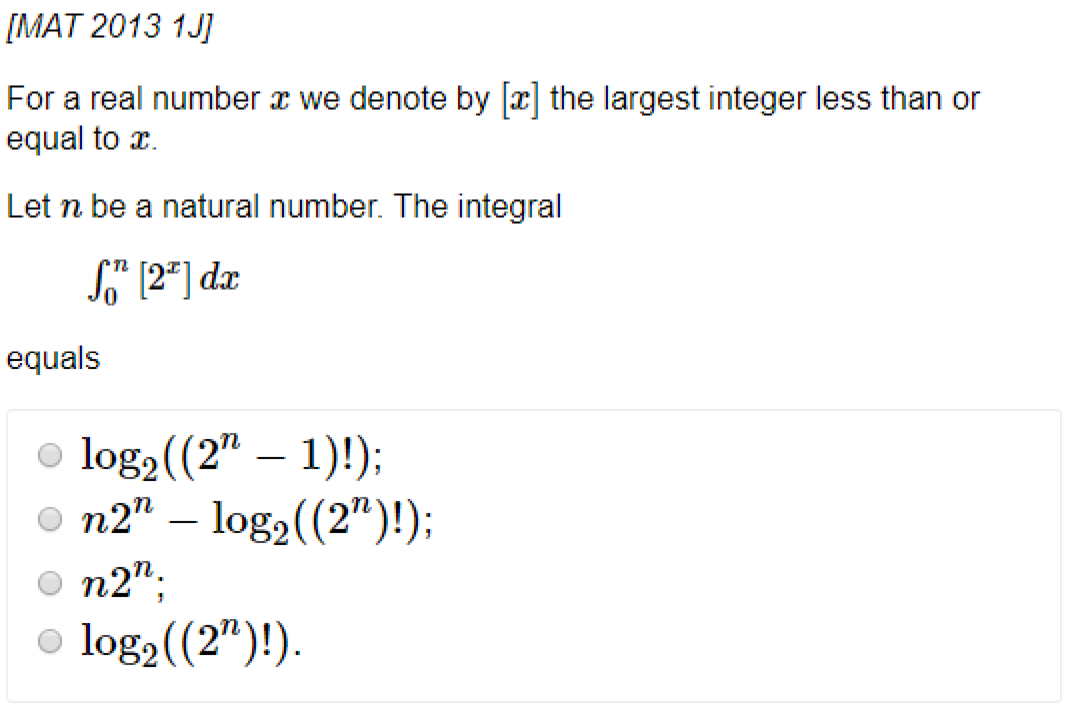
How many distinct solutions does the following equation have?

$$\log_{x^2+2}(4-5x^2-6x^3)=2$$

○ None
○ 1
○ 2
○ 4
○ Infinitely many.

Solution: 2

Question 6



[MAT 2013 1J]

For a real number x we denote by $[x]$ the largest integer less than or equal to x.

Let n be a natural number. The integral

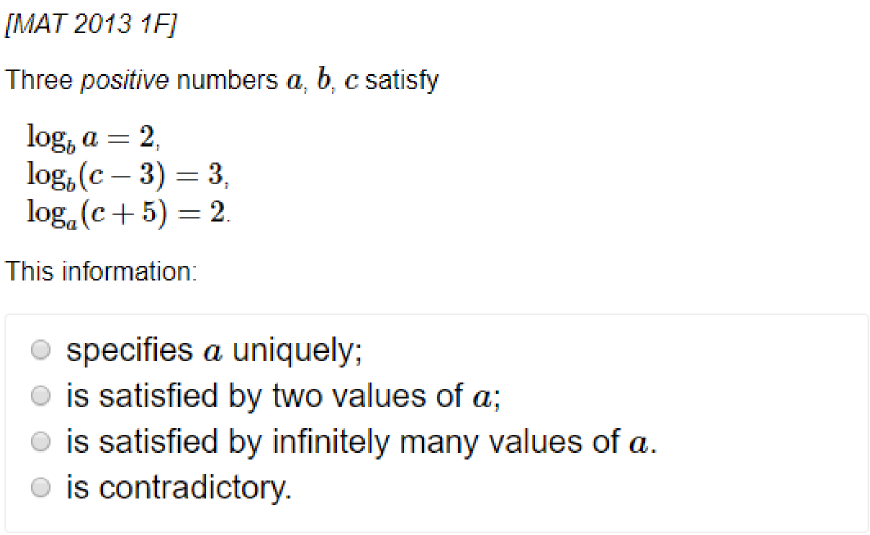
$$\int_0^n [2^x]\,dx$$

equals

○ $\log_2((2^n-1)!)$;
○ $n2^n-\log_2((2^n)!)$;
○ $n2^n$;
○ $\log_2((2^n)!)$.

Solution: Option 2

Question 7



[MAT 2013 1F]

Three *positive* numbers a, b, c satisfy

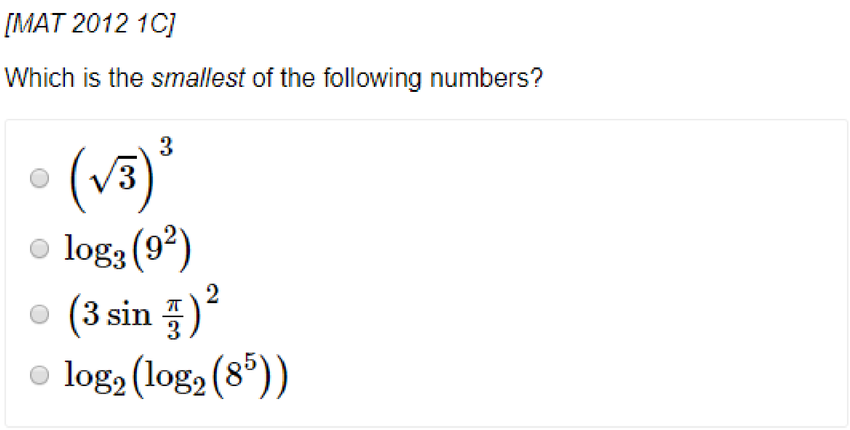
$$\log_b a = 2,$$
$$\log_b (c-3) = 3,$$
$$\log_a (c+5) = 2.$$

This information:

- ○ specifies a uniquely;
- ○ is satisfied by two values of a;
- ○ is satisfied by infinitely many values of a.
- ○ is contradictory.

Solution: Option 1

Question 8



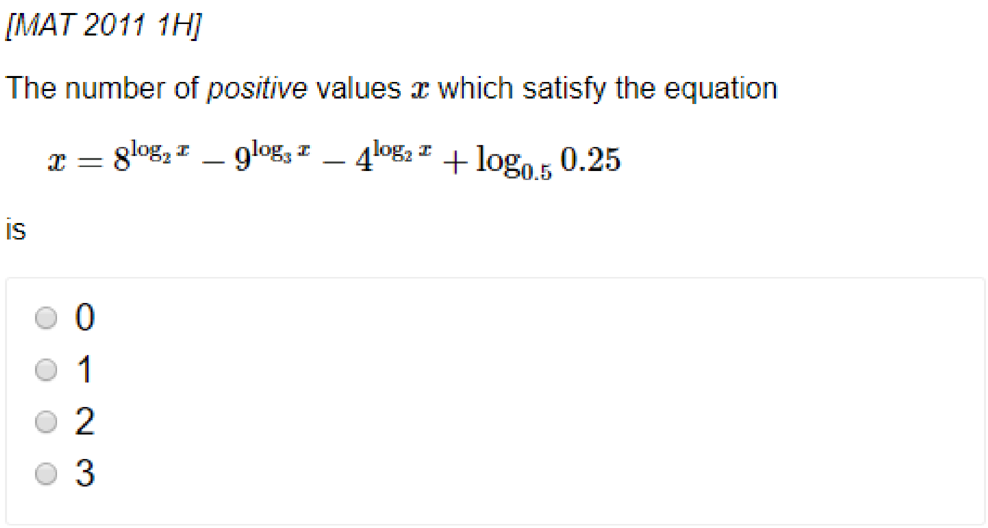
[MAT 2012 1C]

Which is the *smallest* of the following numbers?

- ○ $\left(\sqrt{3}\right)^3$
- ○ $\log_3 (9^2)$
- ○ $\left(3 \sin \frac{\pi}{3}\right)^2$
- ○ $\log_2 (\log_2 (8^5))$

Solution: Option 4

Question 9



[MAT 2011 1H]

The number of *positive* values x which satisfy the equation

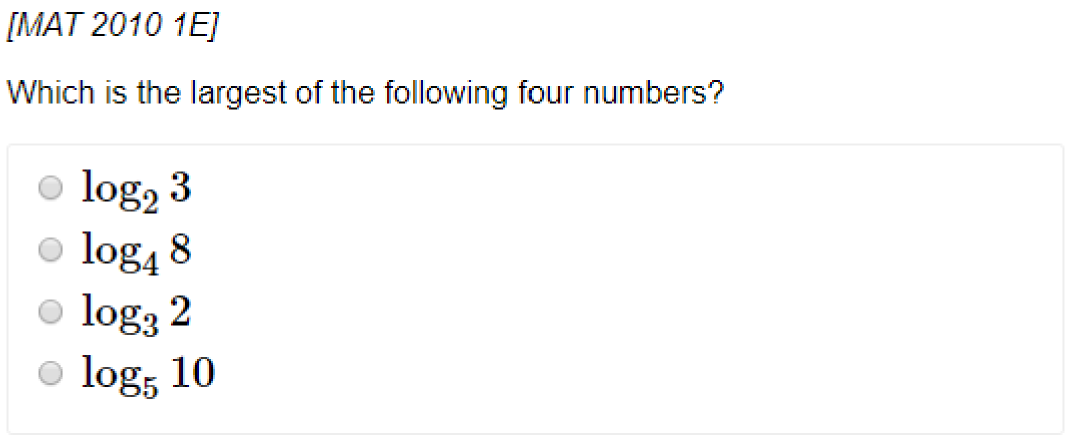
$$x = 8^{\log_2 x} - 9^{\log_3 x} - 4^{\log_2 x} + \log_{0.5} 0.25$$

is

- ○ 0
- ○ 1
- ○ 2
- ○ 3

Solution: 2

Question 10



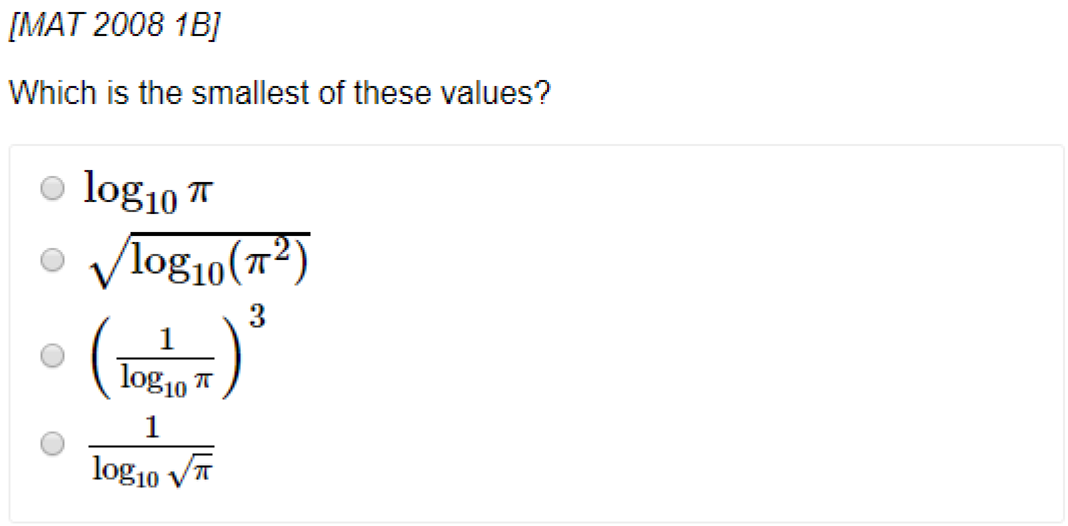
[MAT 2010 1E]

Which is the largest of the following four numbers?

○ $\log_2 3$
○ $\log_4 8$
○ $\log_3 2$
○ $\log_5 10$

Solution: Option 1

Question 11



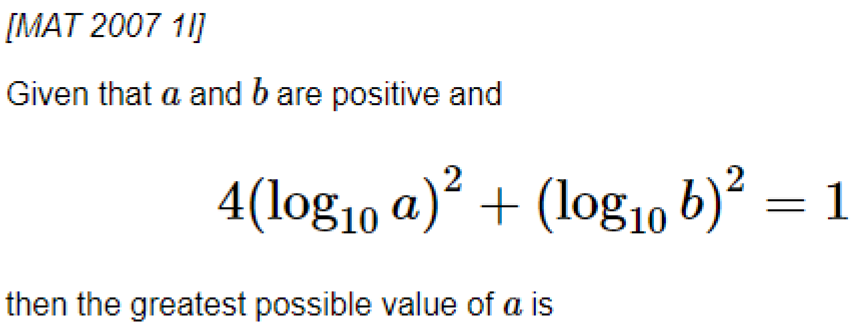
[MAT 2008 1B]

Which is the smallest of these values?

○ $\log_{10} \pi$
○ $\sqrt{\log_{10}(\pi^2)}$
○ $\left(\frac{1}{\log_{10} \pi}\right)^3$
○ $\frac{1}{\log_{10} \sqrt{\pi}}$

Solution: $\log_{10} \pi$

Question 12



[MAT 2007 1I]

Given that a and b are positive and

$$4(\log_{10} a)^2 + (\log_{10} b)^2 = 1$$

then the greatest possible value of a is

Solution: $\sqrt{10}$

Question 13 (AEA 2012 Q5)

[In this question the values of a, x, and n are such that a and x are positive real numbers, with $a > 1, x \neq a, x \neq 1$ and n is an integer with $n > 1$]

Sam was confused about the rules of logarithms and thought that

$$\log_a x^n = (\log_a x)^n \qquad (1)$$

(a) Given that x satisfies statement (1) find x in terms of a and n.

(3)

Sam also thought that

$$\log_a x + \log_a x^2 + ... + \log_a x^n = \log_a x + (\log_a x)^2 + ... + (\log_a x)^n \qquad (2)$$

(b) For $n = 3$, x_1 and x_2 $(x_1 > x_2)$ are the two values of x that satisfy statement (2).

(i) Find, in terms of a, an expression for x_1 and an expression for x_2.

(ii) Find the exact value of $\log_a\left(\frac{x_1}{x_2}\right)$.

(5)

(c) Show that if $\log_a x$ satisfies statement (2) then

$$2(\log_a x)^n - n(n+1)\log_a x + (n^2 + n - 2) = 0$$

(6)

Qu	Scheme	Mark	Notes
(a)	$\log_a x^n = (\log_a x)^n \Rightarrow n\log_a x = (\log_a x)^n$	M1	Use of the power rule to form an equation
	$n = (\log_a x)^{n-1} \Rightarrow \log_a x = n^{\frac{1}{n-1}}$	M1	Attempt root to get an expression for log
	$x = a^{n^{\frac{1}{n-1}}}$ (o.e.)	A1 (3)	
(b) (i)	$(\log_a x)^3 + (\log_a x)^2 - 5\log_a x = 0$ or $(\log_a x)^3 - 6\log_a x + 5 = 0$	M1	Use $n = 3$ to get either
	Let $u = \log_a x$ and solve $u^2 + u - 5 = 0 \rightarrow u = \frac{-1 \pm \sqrt{21}}{2}$	M1	Attempt to solve relevant quadratic.
	$x_1 = a^{\frac{-1+\sqrt{21}}{2}}, x_2 = a^{\frac{-1-\sqrt{21}}{2}}$	A1	
(b)(ii)	$\log_a\left(\frac{x_1}{x_2}\right) = \log_a x_1 - \log_a x_2 = \frac{-1+\sqrt{21}}{2} - \frac{-1-\sqrt{21}}{2}$	M1	Use log$x$ - log$y$ rule and attempt to sub values for x
	$= \sqrt{21}$	A1 (5)	
(c)	LHS = $\log_a x(1 + 2 + ... + n)$	M1	Attempt to use power rule on all of LHS
	$= \log_a x\left(\frac{n(n+1)}{2}\right)$	A1	
	RHS = $\frac{\log_a x\left[(\log_a x)^n - 1\right]}{\log_a x - 1}$	M1	Identify and attempt sum of GP
		A1	
	Equate: $\cancel{\log_a x}\left(\frac{n(n+1)}{2}\right) = \frac{\cancel{\log_a x}\left[(\log_a x)^n - 1\right]}{\log_a x - 1}$	dM1	Equate and attempt to simplify to given answer. Dep on bothMs
	$\log_a x[n(n+1)] - (n^2 + n) = 2(\log_a x)^n - 2$ leading to answer	A1 (6)	cso
		[14]	

Question 14 (AEA 2010 Q1)

(a) Solve the equation

$$\sqrt{(3x+16)}=3+\sqrt{(x+1)}$$

(5)

(b) Solve the equation

$$\log_3(x-7)-\tfrac{1}{2}\log_3 x=1-\log_3 2$$

(7)

1(a)	$3x+16=9+x+1+6\sqrt{x+1}$	M1	Initial squaring -both sides
	$3+x=3\sqrt{x+1}$ (o.e.)	A1	Correct collecting of terms
	$9+6x+x^2=9(x+1)$ or $y=\sqrt{x+1}\rightarrow 3$TQ in y	M1	2nd squaring o.e.
	$x^2-3x=0$ or $(y-2)(y-1)=0$	A1	Both values
	x = 0 or 3	B1 (5)	(S+ for checking values)
(b)	$\frac{1}{2}\log_3 x=\log_3\sqrt{x}$	B1	For use of $n\log x$ rule
	$\log_3(x-7)-\log_3\sqrt{x}=\log_3\frac{x-7}{\sqrt{x}}$	M1	For reducing xs to a single log
	So $2x-14=3\sqrt{x}$ (o.e. all x terms on same line)	M1A1	M1 for getting out of logs A1 for correct equation
	$2\left(\sqrt{x}\right)^2-3\sqrt{x}-14=0$	M1	Attempt to solve suitable 3TQ in x or $\sqrt{x}$
	$\left(2\sqrt{x}-7\right)\left(\sqrt{x}+2\right)=0$		
	$\sqrt{x}=\frac{7}{2}$ or -2	A1	Either solution for $\sqrt{x}$ or x. Must be rational a/b
	$x=\frac{49}{4}$	A1 (7)	49/4 oe only (S+ for clear reason for rejecting $x=4$)
		[12]	

Question 15 (AEA 2008 Q5)

(i) Anna, who is confused about the rules for logarithms, states that

$$\left(\log_3 p\right)^2=\log_3\left(p^2\right)$$

and $$\log_3(p+q)=\log_3 p+\log_3 q.$$

However, there is a value for p and a value for q for which both statements are correct.

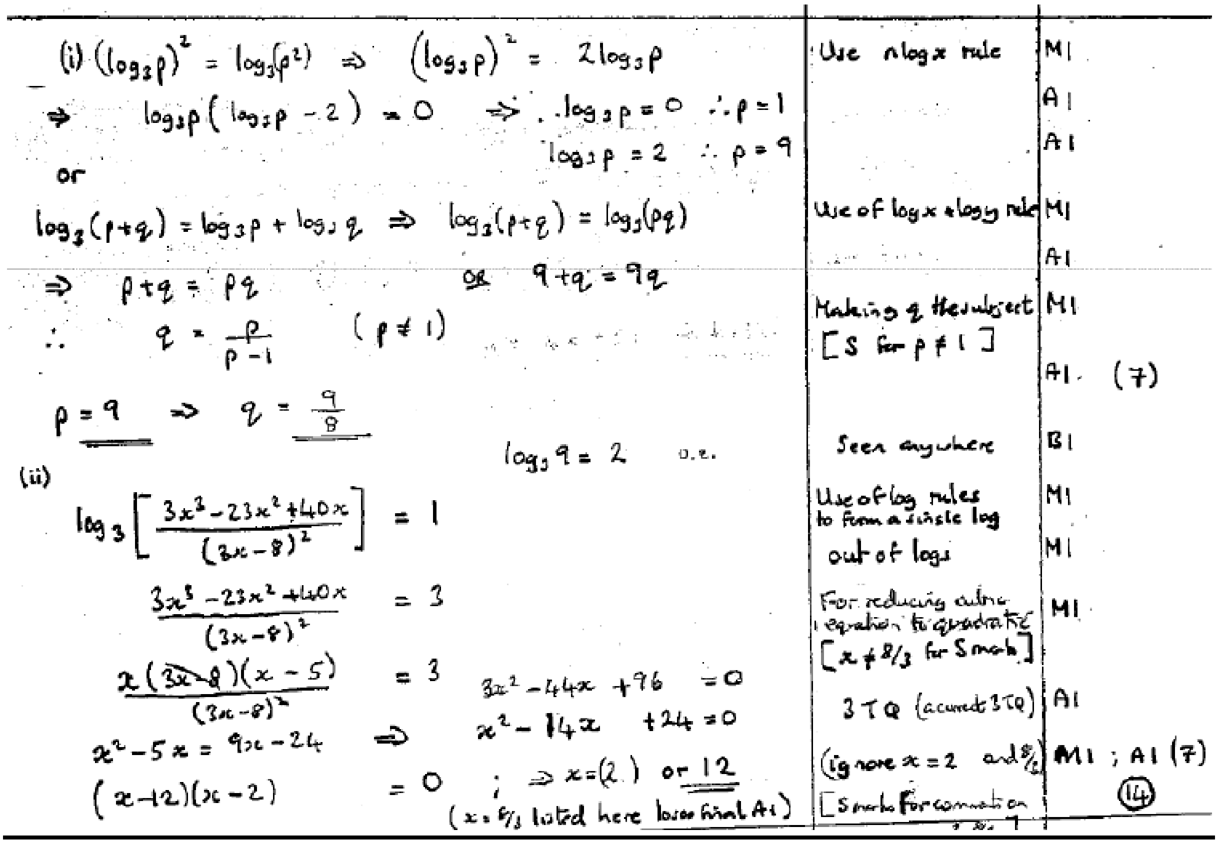
Find the value of p and the value of q.

(7)

(ii) Solve

$$\frac{\log_3(3x^3-23x^2+40x)}{\log_3 9}=0.5+\log_3(3x-8).$$

(7)



(i) $(\log_3 p)^2 = \log_3(p^2) \Rightarrow (\log_3 p)^2 = 2\log_3 p$	Use $n\log x$ rule	M1
$\Rightarrow \log_3 p(\log_3 p - 2) = 0 \Rightarrow \log_3 p = 0 \therefore p = 1$		A1
$\log_3 p = 2 \therefore p = 9$		A1
or		
$\log_3(p+q) = \log_3 p + \log_3 q \Rightarrow \log_3(p+q) = \log_3(pq)$	Use of $\log x + \log y$ rule	M1
		A1
$\Rightarrow p+q = pq$ OR $9+q = 9q$	Making q the subject	M1
$\therefore q = \frac{p}{p-1}$ $(p \neq 1)$	[S for $p \neq 1$]	A1 (7)
$\underline{p=9} \Rightarrow q = \underline{\frac{9}{8}}$		
$\log_3 9 = 2$ o.e.	Seen anywhere	B1
(ii) $\log_3\left[\frac{3x^3-23x^2+40x}{(3x-8)^2}\right] = 1$	Use of log rules to form a single log	M1
	out of logs	M1
$\frac{3x^3-23x^2+40x}{(3x-8)^2} = 3$	For reducing cubic equation to quadratic [$x \neq 8/3$ for S mark]	M1
$\frac{x\cancel{(3x-8)}(x-5)}{(3x-8)^2} = 3$ $\quad 3x^2 - 44x + 96 = 0$	3TQ (accurate 3TQ)	A1
$x^2 - 5x = 9x - 24 \Rightarrow x^2 - 14x + 24 = 0$	(ignore $x = 2$ and $8/3$)	M1 ; A1 (7)
$(x-12)(x-2) = 0$; $\Rightarrow x = (2)$ or $\underline{12}$ ($x = 8/3$ listed here loses final A1)	[S marks for connection	⑭

Question 16 (AEA 2006 Q3)

Given that $x > y > 0$,

(*a*) by writing $\log_y x = z$, or otherwise, show that $\log_y x = \frac{1}{\log_x y}$.

(2)

(*b*) Given also that $\log_x y = \log_y x$, show that $y = \frac{1}{x}$.

(2)

(*c*) Solve the simultaneous equations

$$\log_x y = \log_y x,$$

$$\log_x (x - y) = \log_y (x + y).$$

(7)

$\log_y x = z \quad \therefore \; x = y^z$ M1

(a) $\therefore \; y = x^{1/z} \Rightarrow \log_x y = \frac{1}{z} = \frac{1}{\log_y x}$ (cont of logs)

or $\log_x x = \log_x y^z = z \log_x y = 1$

$\therefore \; \log_x y = \frac{1}{\log_y x}$ (Answer) A1 c.s.o (2)

(b) $\log_x y = \log_y x = \frac{1}{\log_x y} \Rightarrow (\log_x y)^2 = 1$

$\therefore \; \log_x y = \pm 1$ M1

$\log_x y \neq 1 \; \because \; x \neq y. \quad \therefore \; \log_x y = -1$

$\therefore \; \underline{y = \frac{1}{x}}$ A1 c.s.o (2)

(c) First equation $\Rightarrow y = 1/x$

Second equation $\Rightarrow \log_x \left(x - \frac{1}{x}\right) = \log_{\frac{1}{x}} \left(x + \frac{1}{x}\right) = z$ M1

$\therefore \; x^z = x - \frac{1}{x} \quad ; \; \left(\frac{1}{x}\right)^z = x + \frac{1}{x}$

$\therefore \; x^z \left(\frac{1}{x}\right)^z = 1 = \left(x - \frac{1}{x}\right)\left(x + \frac{1}{x}\right)$ (eliminate z) M1

$\Rightarrow x^2 = (x^2 - 1)(x^2 + 1)$

$\Rightarrow x^4 - x^2 - 1 = 0$ (quadratic) M1, A1

$x^2 = (1 \pm \sqrt{5})/2$

$x^2 > 0$, hence $x^2 = \frac{1 + \sqrt{5}}{2}$ (quad formula + $x^2 > 0$) M1

$\underline{x = \sqrt{\frac{1+\sqrt{5}}{2}}}$ (valid +ve root) A1

$y = \frac{1}{x} = \underline{\sqrt{\frac{2}{1+\sqrt{5}}}} \quad \left(\text{or } \sqrt{\frac{\sqrt{5}-1}{2}}\right)$ A1$\checkmark$

(7)

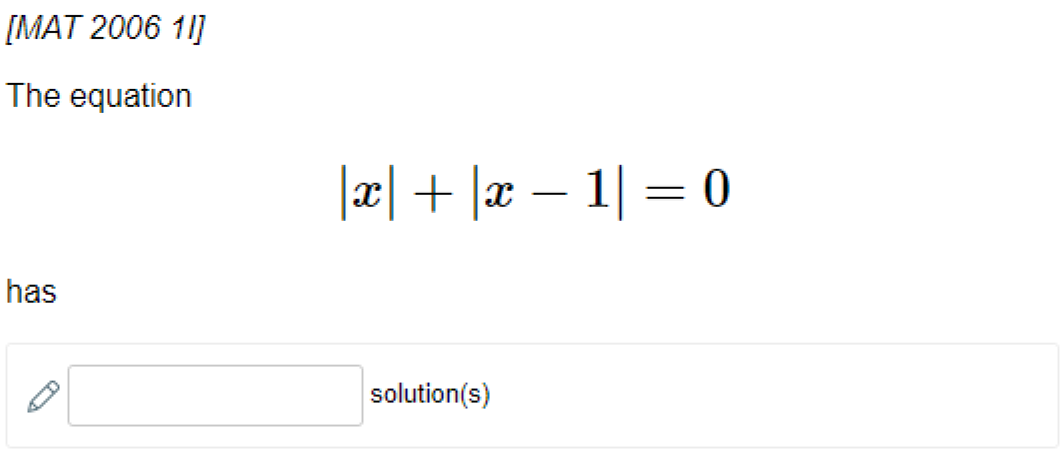
2 YEAR 2

2.1 ALGEBRAIC METHODS (PROOF BY CONTRADICTION, PARTIAL FRACTIONS)

(No questions available)

2.2 FUNCTIONS & GRAPHS (MODULUS, MAPPING, COMPOSITE, INVERSE, SOLVING MODULUS EQUATIONS)

Question 1



[MAT 2006 1I]

The equation

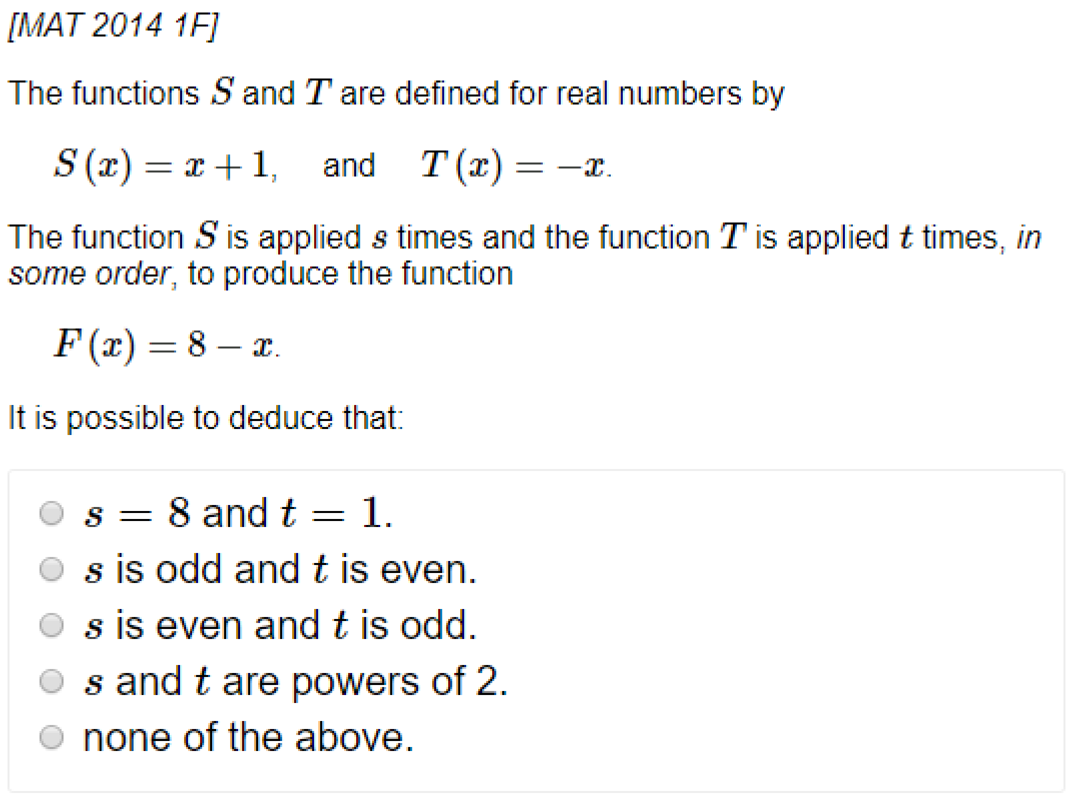
$$|x| + |x - 1| = 0$$

has

solution(s)

Solution: 0

Question 2



[MAT 2014 1F]

The functions S and T are defined for real numbers by

$$S(x) = x + 1, \quad \text{and} \quad T(x) = -x.$$

The function S is applied s times and the function T is applied t times, *in some order*, to produce the function

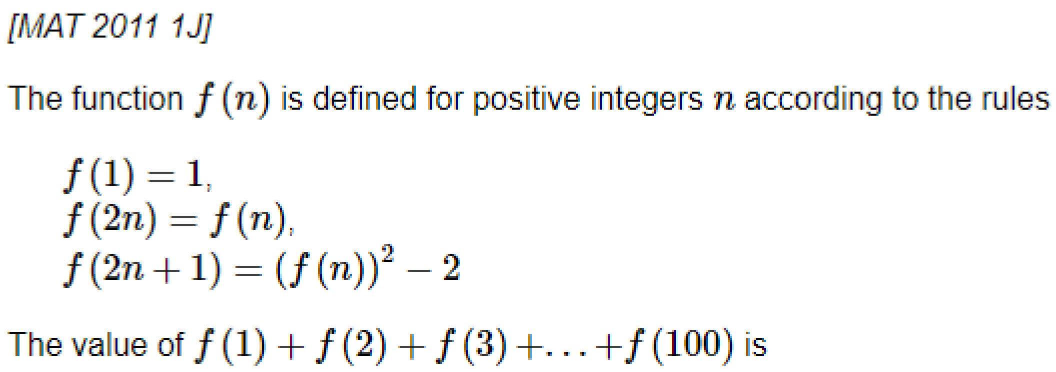
$$F(x) = 8 - x.$$

It is possible to deduce that:

- ○ $s = 8$ and $t = 1$.
- ○ s is odd and t is even.
- ○ s is even and t is odd.
- ○ s and t are powers of 2.
- ○ none of the above.

Solution: $s = 8$ and $t = 1$

Question 3



[MAT 2011 1J]

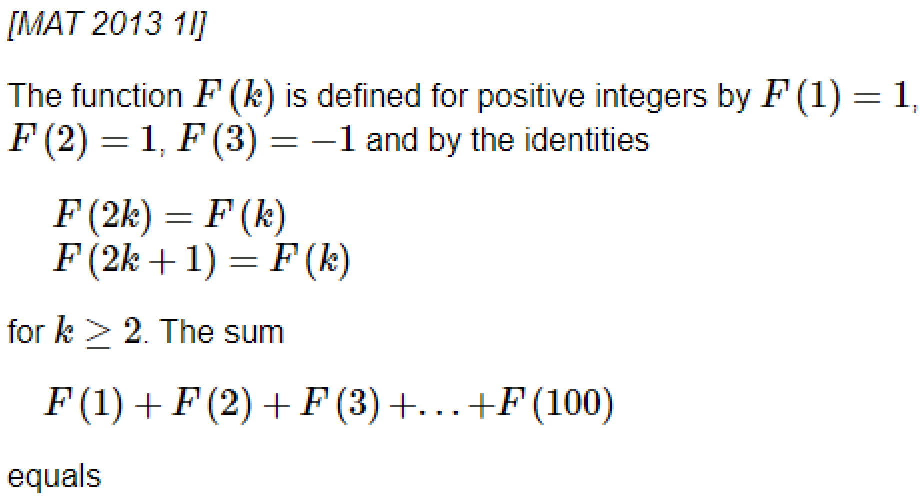
The function $f(n)$ is defined for positive integers n according to the rules

$$f(1) = 1,$$
$$f(2n) = f(n),$$
$$f(2n+1) = (f(n))^2 - 2$$

The value of $f(1) + f(2) + f(3) + \ldots + f(100)$ is

Solution: -86

Question 4



[MAT 2013 1I]

The function $F(k)$ is defined for positive integers by $F(1) = 1$, $F(2) = 1$, $F(3) = -1$ and by the identities

$$F(2k) = F(k)$$
$$F(2k+1) = F(k)$$

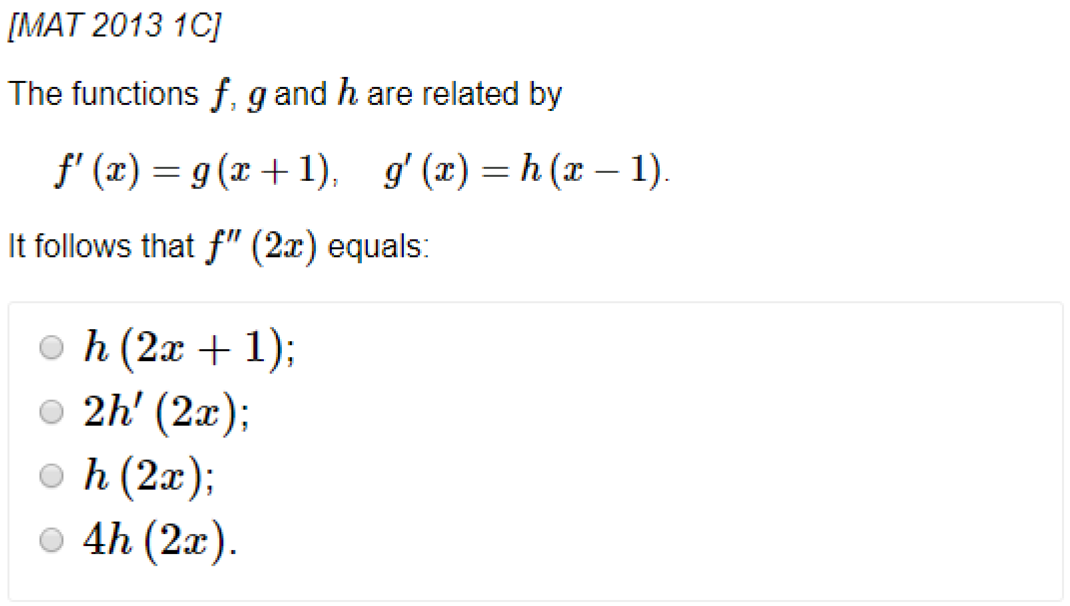
for $k \geq 2$. The sum

$$F(1) + F(2) + F(3) + \ldots + F(100)$$

equals

Solution: 28

Question 5



[MAT 2013 1C]

The functions f, g and h are related by

$$f'(x) = g(x+1), \quad g'(x) = h(x-1).$$

It follows that $f''(2x)$ equals:

- ○ $h(2x+1)$;
- ○ $2h'(2x)$;
- ○ $h(2x)$;
- ○ $4h(2x)$.

Solution: $h(2x)$

Question 6

[MAT 2010 1G]

The function f, defined for whole positive numbers, satisfies $f(1) = 1$ and also the rules

$$f(2n) = 2f(n)$$
$$f(2n+1) = 4f(n)$$

for all values of n. How many numbers n satisfy $f(n) = 16$?

Solution: 5

Question 7 (AEA 2013 Q7)

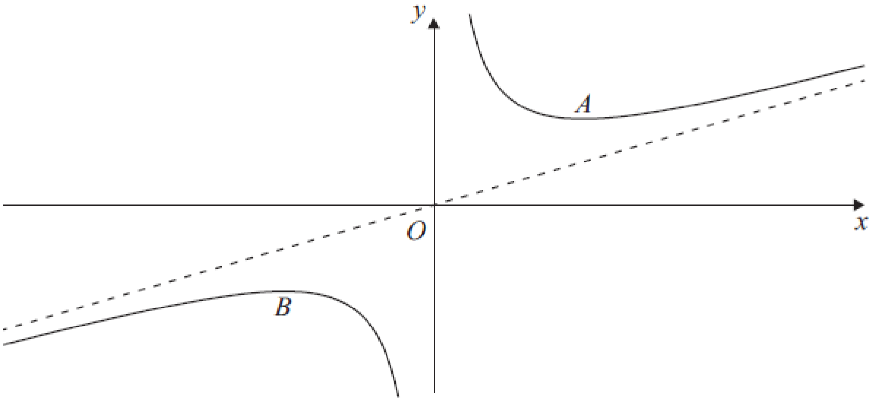


Figure 1

Figure 1 shows a sketch of the curve C_1 with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{x}{3} + \frac{12}{x} \qquad x \neq 0$$

The lines $x = 0$ and $y = \frac{x}{3}$ are asymptotes to C_1. The point A on C_1 is a minimum and the point B on C_1 is a maximum.

(a) Find the coordinates of A and B.

(4)

There is a normal to C_1, which does not intersect C_1 a second time, that has equation $x = k$, where $k > 0$.

(b) Write down the value of k.

(1)

The point $P\ (\alpha, \beta)$, $\alpha > 0$ and $\alpha \neq k$, lies on C_1. The normal to C_1 at P does not intersect C_1 a second time.

(c) Find the value of α, leaving your answer in simplified surd form.

(5)

(d) Find the equation of this normal.

(3)

The curve C_2 has equation $y = |\mathrm{f}(x)|$

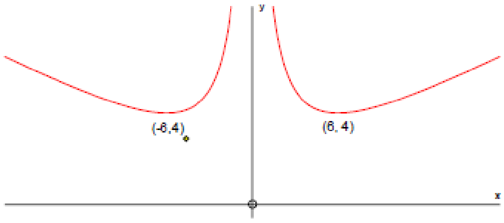
(e) Sketch C_2 stating the coordinates of any turning points and the equations of any asymptotes.

(4)

The line with equation $y = mx + 1$ does not touch or intersect C_2.

(f) Find the set of possible values for m.

(5)

Question	Scheme	Marks	Notes
(a)	$f'(x)=\frac{1}{3}-12x^{-2}$	M1	Some correct diff
	$f'(x)=0 \Rightarrow x^2=36$	M1	f '(x) =0 to give x^2=..
	So *A* (6, 4) and *B* $(-6,-4)$ [1st A1 for ± 6 or (6, 4)]	A1A1 (4)	2nd A1 is cso
(b)	$k=6$ (Allow $k=\pm 6$)	B1ft (1)	
(c)	Grad of normal = $\frac{1}{3}$, so gradient of tangent must be -3	B1M1	M1 for perp. rule
S+ for B1 comment	So $-3=\frac{1}{3}-12x^{-2}$ $\left[f'(x)=-3 \text{ or } \frac{-1}{f'(x)}=\frac{1}{3}\right]$	dM1	Form a suitable eqn using their f′(*x*)
	$x^2=\frac{36}{10}$ so $(\alpha=)\frac{6}{\sqrt{10}}$ or $\frac{3}{5}\sqrt{10}$ or $3\sqrt{\frac{2}{5}}$	dM1 A1 (5)	Solving suitable eqn $p\sqrt{q}$ where *p* or *q* is an integer
(d)	y coord: $\beta=\frac{\sqrt{10}}{5}+\frac{12\sqrt{10}}{6}=2.2\sqrt{10}$ or $\frac{11}{5}\sqrt{10}$	M1	Attempt *y* coord
	Equation of normal is: $y-\beta=\frac{1}{3}(x-\alpha)$	M1	ft their α and β Must be values and $m=\frac{1}{3}$
	i.e. $y=\frac{1}{3}x+2\sqrt{10}$ (o.e.)	A1 (3)	
(e)	 Shape (6, 4); (-6, 4) <u>Asymptotes</u> $x=0$, $y=\pm\frac{1}{3}x$	B1 B1ft B1B1 (4)	Both branches Follow through their *A* and *B* -1 each omission $y=\left\|\frac{x}{3}\right\|$ is OK
(f)	If intersect then line = curve gives: $(3m-1)x^2+3x-36=0$	M1	Attempt line = curve $\rightarrow$3TQ
S+ for comment	Disciminant <0 gives: $9<4\times(3m-1)(-36)$	M1	Correct use of discr leading to ineq in *m*
	Solving : $48m<15$, so $m<\frac{5}{16}$	M1 A1	Solving to $m<k$ A1 for $k=\frac{5}{16}$ (o.e.)
S+ for comment on $m>$ ...	From sketch : $-\frac{5}{16}<m<\frac{5}{16}$	A1 (5)	Both [Allow M1M1M1 for MR of *l* for 1]
ALT (f)	Tangent at $\left(\delta, \frac{\delta}{3}+\frac{12}{\delta}\right)$ goes through (0, 1), gradient = m = f ′ (δ)		Use of limiting case: gradient of chord = gradient of tangent (= gradient of line)
	Leads to equation: $\frac{1}{3}-\frac{12}{\delta^2}=\frac{\frac{\delta}{3}+\frac{12}{\delta}-1}{\delta}$	M1	
	$\frac{\delta^2-36}{3\delta^2}=\frac{\delta^2+36-3\delta}{3\delta^2} \Rightarrow 3\delta=72$ or $\delta=24$	M1	Solve for δ
	$m=\frac{1}{3}-\frac{12}{\delta^2}=\frac{5}{16}$ etc		Then as above

Question 8 (AEA 2012 Q1)

The function f is given by

$$f(x) = x^2 - 2x + 6, \qquad x \geqslant 0$$

(a) Find the range of f.

(3)

The function g is given by

$$g(x) = 3 + \sqrt{(x + 4)}, \qquad x \geqslant 2$$

(b) Find gf(x).

(2)

(c) Find the domain and range of gf.

(3)

Qu	Scheme	Mark	Notes
(a)	$x^2-2x+6=(x-1)^2+5$ <u>or</u> $2x-2=0$	M1	Differentiating or complete the square
	Sketch or work to show min at (1, 5)	A1	
	Range **f $\ge$ 5** (Accept $y \ge 5$) (Answer only 3/3)	A1 **(3)**	$x \ge 5$ can score M1A1A0
(b)	$\text{gf}(x) = 3+\sqrt{x^2-2x+6+4}, = 3+\sqrt{x^2-2x+10}$	M1,A1	
		(2)	
(c)	$\text{gf}(1)$ or $3+\sqrt{"5"+4}$	M1	Clear attempt to find gf(1) or correct express'
	Range of **gf $\ge$ 6**	A1	
	Domain = domain of f = **$x \ge 0$**	B1 **(3)**	
		[8]	

Question 9 (AEA 2009 Q1)

(a) On the same diagram, sketch

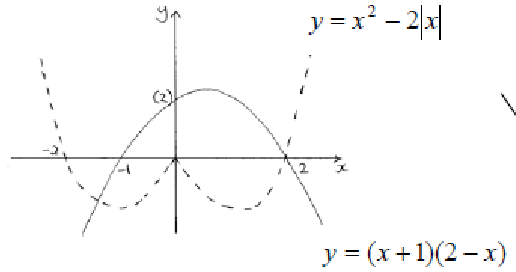
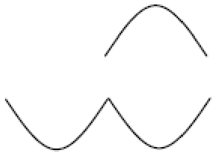
$$y = (x + 1)(2 - x) \quad \text{and} \quad y = x^2 - 2|x|.$$

Mark clearly the coordinates of the points where these curves cross the coordinate axes.

(3)

(b) Find the x-coordinates of the points of intersection of these two curves.

(5)

(a)	  -2, -1, 2, (0,2)	B1 B1 B1 (3)	Don't insist on labels
(b)	One intersection at $x = 2$	B1	
	Second at $(x+1)(2-x) = x(x+2)$	M1	Attempt correct equation Must be $x + 2$ on RHS
	$(0 =)\ 2x^2 + x - 2$	A1	Correct 3TQ
	$x = \dfrac{-1 \pm \sqrt{1+16}}{4}$, since root is in (-2, -1) $x = \dfrac{-1-\sqrt{17}}{4}$	M1 A1 cso (5)	Solving Must choose -
		[8]	

Question 10 (AEA 2008 Q6)

$$f(x)=\frac{ax+b}{x+2};\quad x\in\mathbb{R},\ x\neq -2,$$

where a and b are constants and $b > 0$.

(a) Find $f^{-1}(x)$.

(2)

(b) Hence, or otherwise, find the value of a so that $ff(x) = x$.

(2)

The curve C has equation $y = f(x)$ and $f(x)$ satisfies $ff(x) = x$.

(c) On separate axes sketch

(i) $y = f(x)$,

(3)

(ii) $y = f(x - 2) + 2$.

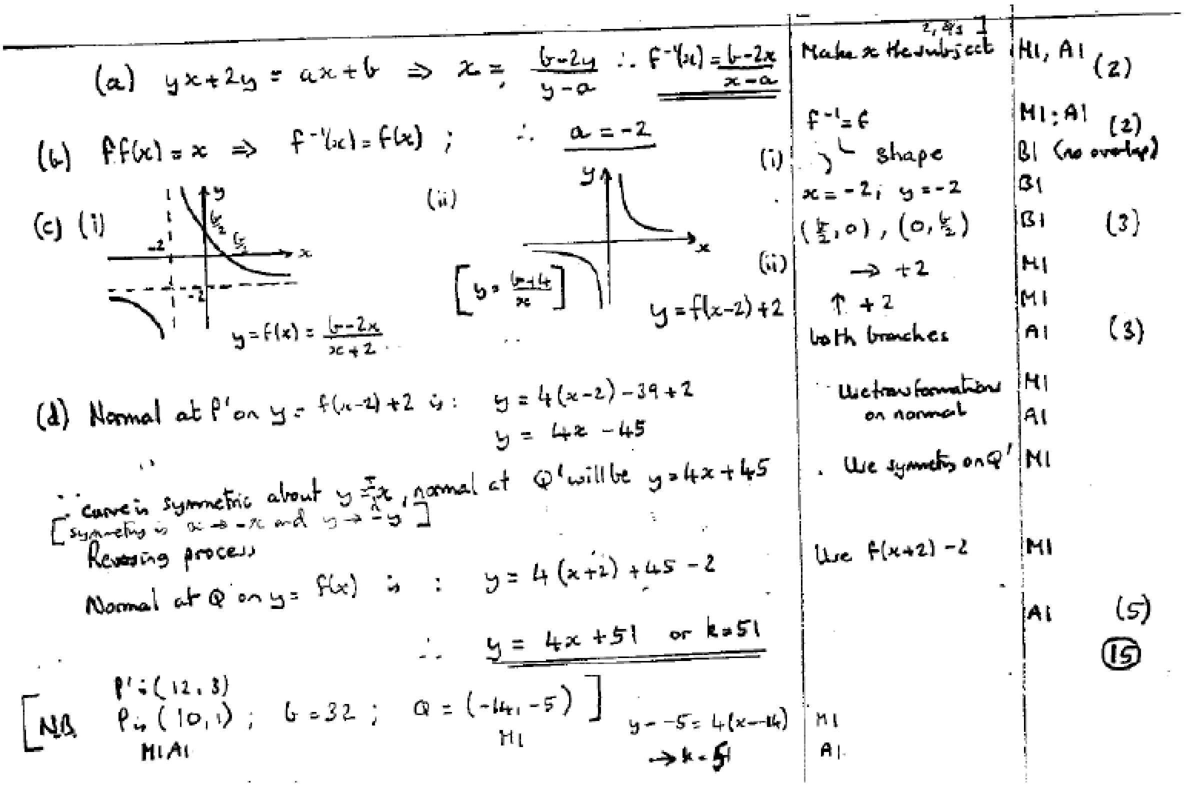
(3)

On each sketch you should indicate the equations of any asymptotes and the coordinates, in terms of b, of any intersections with the axes.

The normal to C at the point P has equation $y = 4x - 39$. The normal to C at the point Q has equation $y = 4x + k$, where k is a constant.

(d) By considering the images of the normals to C on the curve with equation $y = f(x - 2) + 2$, or otherwise, find the value of k.

(5)



(a) $yx+2y=ax+b \Rightarrow x=\frac{b-2y}{y-a}$ ∴ $f^{-1}(x)=\frac{b-2x}{x-a}$	Make x the subject	M1, A1 (2)
(b) $ff(x)=x \Rightarrow f^{-1}(x)=f(x)$; ∴ $a=-2$	$f^{-1}=f$	M1; A1 (2)
(c) (i) $y=f(x)=\frac{b-2x}{x+2}$	(i) shape	B1 (no overlap)
	$x=-2$; $y=-2$	B1
	$(\frac{b}{2},0)$, $(0,\frac{b}{2})$	B1 (3)
(ii) $\left[y=\frac{b+4}{x}\right]$ $y=f(x-2)+2$	(ii) $\rightarrow +2$	M1
	↑ $+2$	M1
	both branches	A1 (3)
(d) Normal at P' on $y=f(x-2)+2$ is: $y=4(x-2)-39+2$ $y=4x-45$	Use transformations on normal	M1 A1
∴ curve is symmetric about $y=\pm x$, normal at Q' will be $y=4x+45$ [symmetry is $x\rightarrow -x$ and $y\rightarrow -y$]	Use symmetry on Q'	M1
Reversing process Normal at Q on $y=f(x)$ is : $y=4(x+2)+45-2$	Use $f(x+2)-2$	M1
∴ $y=4x+51$ or $k=51$		A1 (5) ⑮
[NB $P':(12,3)$ P is $(10,1)$; $b=32$; $Q=(-14,-5)$] M1A1 M1 $y - -5 = 4(x - -14)$ $\rightarrow k=51$	M1 A1	

Question 11 (AEA 2005 Q6)

Figure 1

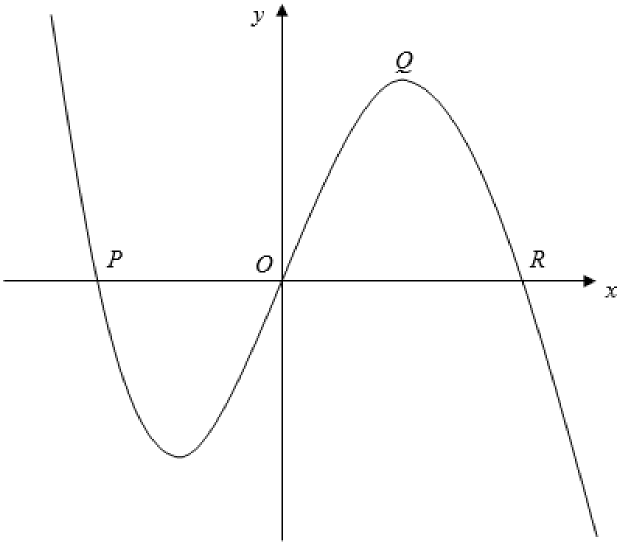


Figure 1 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = x(12 - x^2).$$

The curve cuts the x-axis at the points P, O and R, and Q is the maximum point.

(*a*) Find the coordinates of the points P, Q and R. **(4)**

(*b*) Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2x)$,
(ii) $y = \mathrm{f}(|x| + 1)$,

indicating on each sketch the coordinates of any maximum points and the intersections with the x-axis. **(6)**

Figure 2

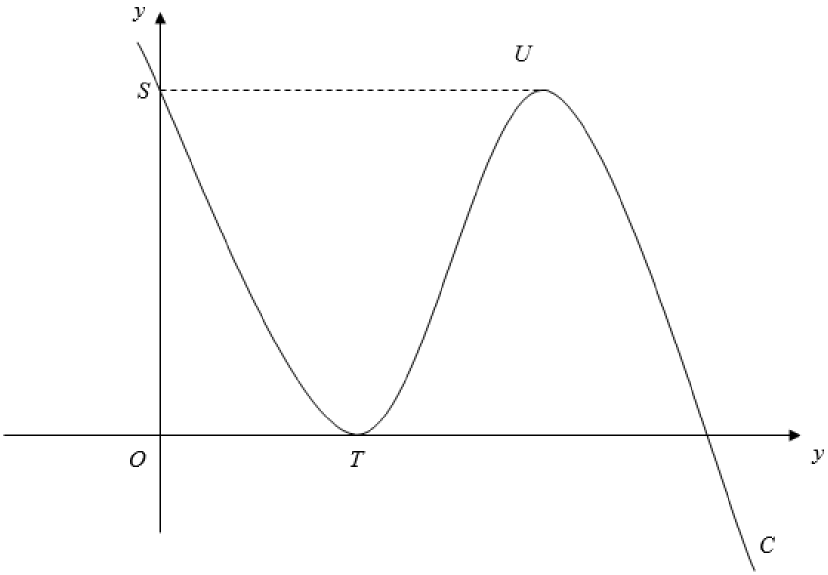


Figure 2 shows a sketch of part of the curve C, with equation $y = \mathrm{f}(x - v) + w$, where v and w are constants. The x-axis is a tangent to C at the minimum point T, and C intersects the y-axis at S. The line joining S to the maximum point U is parallel to the x-axis.

(*c*) Find the value of v and the value of w and hence find the roots of the equation

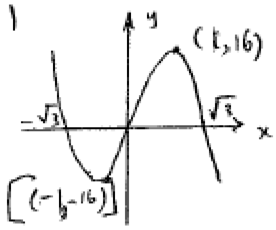
$$\mathrm{f}(x - v) + w = 0.$$

(9)

(a) $f = 0$, $x^2 = 12$ $\therefore$ R and P are $\underline{(\pm\sqrt{12}, 0)}$ B1, B1

$f' = 12 - 3x^2$ $f' = 0$, $x^2 = 4$; so $\underline{Q = (2, 16)}$ M1, A1 (4

(b) (i)

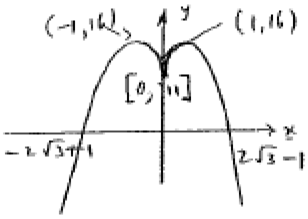


shape, symmetry about O B1

Any one of P′, R′, Q′ M1

all correct A1 √

(ii)



shape, symmetry about axis B1

$(2\sqrt{3}-1, 0)$ and $(1, 16)$
& meets y-axis above origin } B1

$(-2\sqrt{3}+1, 0)$ and $(-1, 16)$ B1 (6

(c) Min. at $(-2, -16) \Rightarrow (\alpha, 0)$

$\therefore$ curve has "moved up" by 16 $\therefore$ $\underline{w = 16}$ M1, A1

$x = 0 \Rightarrow f(-v) + w = 2 \times 16$ ($\because$ S, U have same y-coordinates) M1

$\therefore f(-v) = 16$

$-v(12 - v^2) = 16$

$v^3 - 12v - 16 = 0$ (correct cubic in v) A1

$(v+2)^2(v-4) = 0$ (finding a root) M1

$\therefore v = -2$ or $v = 4$

Min. has moved from −2 to +ve value, so $v > 0$

$\therefore \underline{v = 4}$ A1

B1

Horizontal movement of 4 so $\underline{T \text{ is } (2, 0)}$

$S = (0, 32)$; $T = (2, 0)$; $U = (6, 32)$ so by symmetry M1, A1

other point of intersection is $\underline{x = 8}$, (9)

2.3 SEQUENCES & SERIES (ARITHMETIC, GEOMETRIC, SIGMA NOTATION, RECURRENCES)

Question 1 (STEP I 2012 Q7)

A sequence of numbers t_0, t_1, t_2, … satisfies

$$t_{n+2} = pt_{n+1} + qt_n \qquad (n \geqslant 0),$$

where p and q are real. Throughout this question, x, y and z are non-zero real numbers.

(i) Show that, if $t_n = x$ for all values of n, then $p + q = 1$ and x can be any (non-zero) real number.

(ii) Show that, if $t_{2n} = x$ and $t_{2n+1} = y$ for all values of n, then $q \pm p = 1$. Deduce that either $x = y$ or $x = -y$, unless p and q take certain values that you should identify.

(iii) Show that, if $t_{3n} = x$, $t_{3n+1} = y$ and $t_{3n+2} = z$ for all values of n, then

$$p^3 + q^3 + 3pq - 1 = 0\,.$$

Deduce that either $p + q = 1$ or $(p - q)^2 + (p + 1)^2 + (q + 1)^2 = 0$. Hence show that either $x = y = z$ or $x + y + z = 0$.

Question 2 (STEP I 2004 Q2)

The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}\,] = 4$ and $[5] = 5$.

(i) Sketch the graph of $y = \sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\displaystyle\int_0^a 2^{[x]}\,\mathrm{d}x$ when a is positive but not an integer.

Solutions: (iii) $\left(2^{[a]} - 1\right) + (a - [a]) \times 2^{[a]}$

Question 3 (STEP I 2004 Q5)

The positive integers can be split into five distinct arithmetic progressions, as shown:

$$\begin{aligned} A:&\ \ 1,\ 6,\ 11,\ 16,\ \ldots\\ B:&\ \ 2,\ 7,\ 12,\ 17,\ \ldots\\ C:&\ \ 3,\ 8,\ 13,\ 18,\ \ldots\\ D:&\ \ 4,\ 9,\ 14,\ 19,\ \ldots\\ E:&\ \ 5, 10,\ 15,\ 20,\ \ldots \end{aligned}$$

Write down an expression for the value of the general term in each of the five progressions. Hence prove that the sum of any term in B and any term in C is a term in E.

Prove also that the square of every term in B is a term in D. State and prove a similar claim about the square of every term in C.

(i) Prove that there are no positive integers x and y such that

$$x^2 + 5y = 243\,723\,.$$

(ii) Prove also that there are no positive integers x and y such that

$$x^4 + 2y^4 = 26\,081\,974\,.$$

Question 4 (STEP I 2004 Q7)

(i) The function $\mathrm{f}(x)$ is defined for $|x| < \frac{1}{5}$ by

$$\mathrm{f}(x) = \sum_{n=0}^{\infty} a_n x^n\,,$$

where $a_0 = 2$, $a_1 = 7$ and $a_n = 7a_{n-1} - 10a_{n-2} = 0$ for $n \geqslant 2$.

Simplify $\mathrm{f}(x) - 7x\mathrm{f}(x) + 10x^2\mathrm{f}(x)$, and hence show that $\mathrm{f}(x) = \dfrac{1}{1-2x} + \dfrac{1}{1-5x}$.

Hence show that $a_n = 2^n + 5^n$.

(ii) The function $\mathrm{g}(x)$ is defined for $|x| < \frac{1}{3}$ by

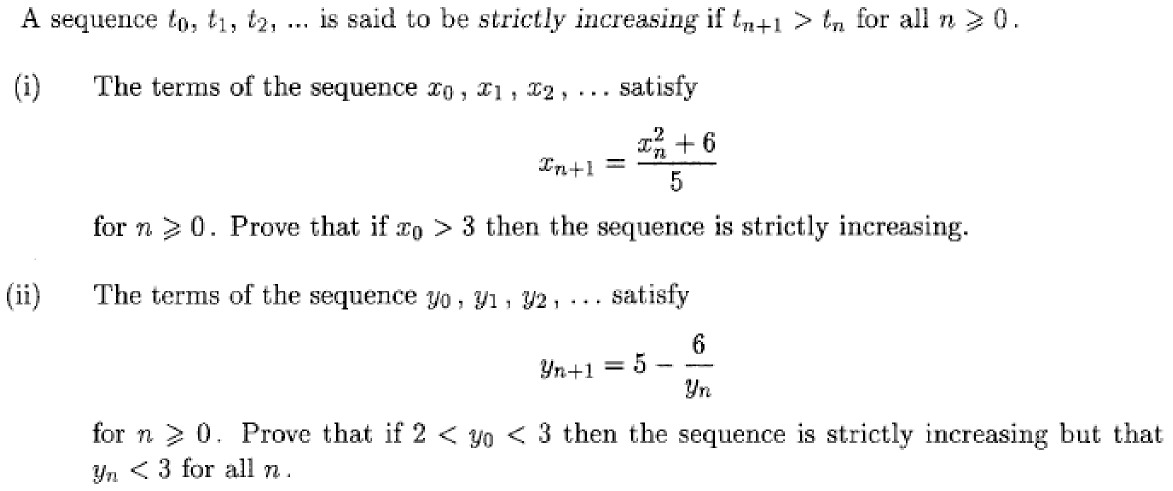
$$\mathrm{g}(x) = \sum_{n=0}^{\infty} b_n x^n\,,$$

where $b_0 = 5$, $b_1 = 10$, $b_2 = 40$, $b_3 = 100$ and $b_n = pb_{n-1} + qb_{n-2}$ for $n \geqslant 2$. Obtain an expression for $\mathrm{g}(x)$ as the sum of two algebraic fractions and determine b_n in terms of n.

Solutions: (ii) $b_n = (-2)^n + 4(3)^n$

Question 5 (STEP I 2004 Q8)

(Note: Knowledge of method of Proof By Induction required for this question)



A sequence t_0, t_1, t_2, ... is said to be *strictly increasing* if $t_{n+1} > t_n$ for all $n \geqslant 0$.

(i) The terms of the sequence x_0 , x_1 , x_2 , . . . satisfy

$$x_{n+1} = \frac{x_n^2+6}{5}$$

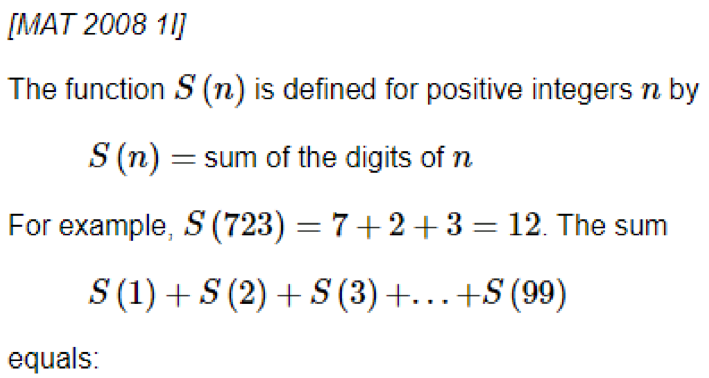
for $n \geqslant 0$. Prove that if $x_0 > 3$ then the sequence is strictly increasing.

(ii) The terms of the sequence y_0 , y_1 , y_2 , . . . satisfy

$$y_{n+1} = 5 - \frac{6}{y_n}$$

for $n \geqslant 0$. Prove that if $2 < y_0 < 3$ then the sequence is strictly increasing but that $y_n < 3$ for all n.

Question 6



[MAT 2008 1I]

The function $S(n)$ is defined for positive integers n by

$$S(n) = \text{sum of the digits of } n$$

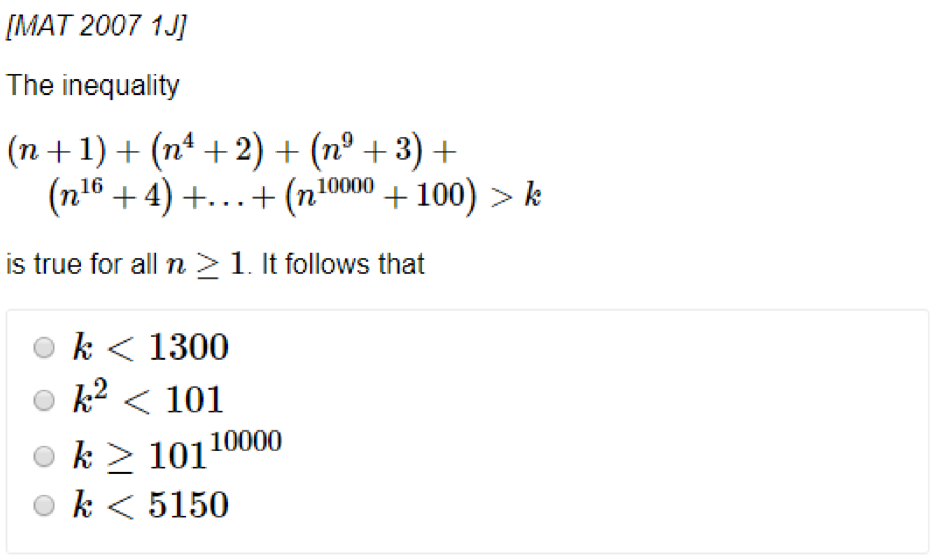
For example, $S(723) = 7+2+3 = 12$. The sum

$$S(1)+S(2)+S(3)+\ldots+S(99)$$

equals:

Solution: 900

Question 7



[MAT 2007 1J]

The inequality

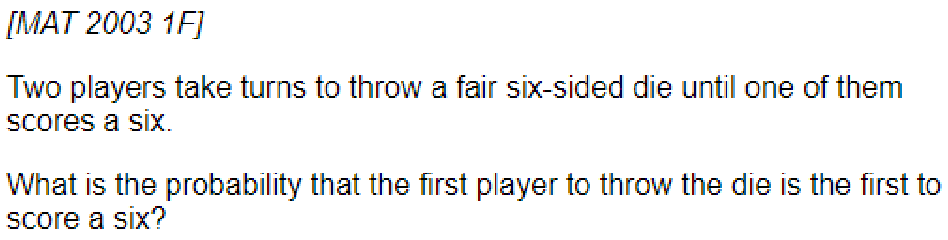
$$(n+1)+(n^4+2)+(n^9+3)+\\(n^{16}+4)+\ldots+(n^{10000}+100) > k$$

is true for all $n \geq 1$. It follows that

○ $k < 1300$
○ $k^2 < 101$
○ $k \geq 101^{10000}$
○ $k < 5150$

Solution: $k < 5150$

Question 8



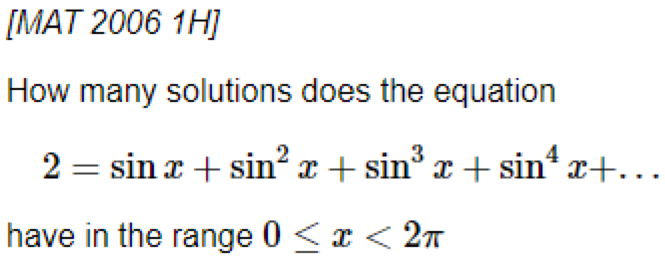
[MAT 2003 1F]

Two players take turns to throw a fair six-sided die until one of them scores a six.

What is the probability that the first player to throw the die is the first to score a six?

Solution: $\frac{6}{11}$

Question 9



[MAT 2006 1H]

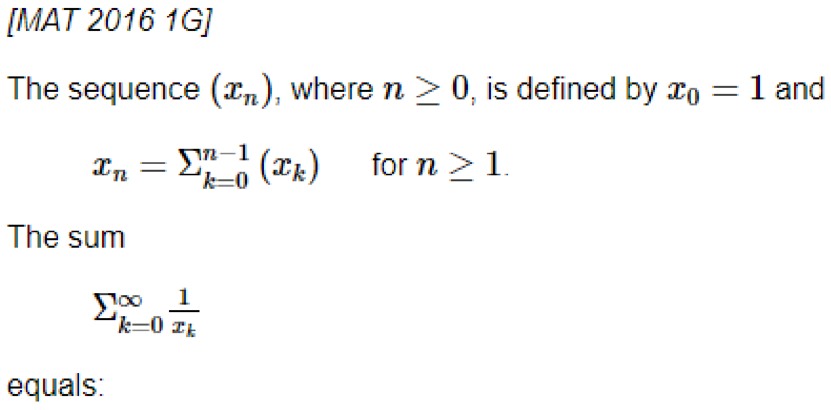
How many solutions does the equation

$$2 = \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \dots$$

have in the range $0 \le x < 2\pi$

Solution: 2 solutions

Question 10



[MAT 2016 1G]

The sequence (x_n), where $n \ge 0$, is defined by $x_0 = 1$ and

$$x_n = \Sigma_{k=0}^{n-1}(x_k) \quad \text{for } n \ge 1.$$

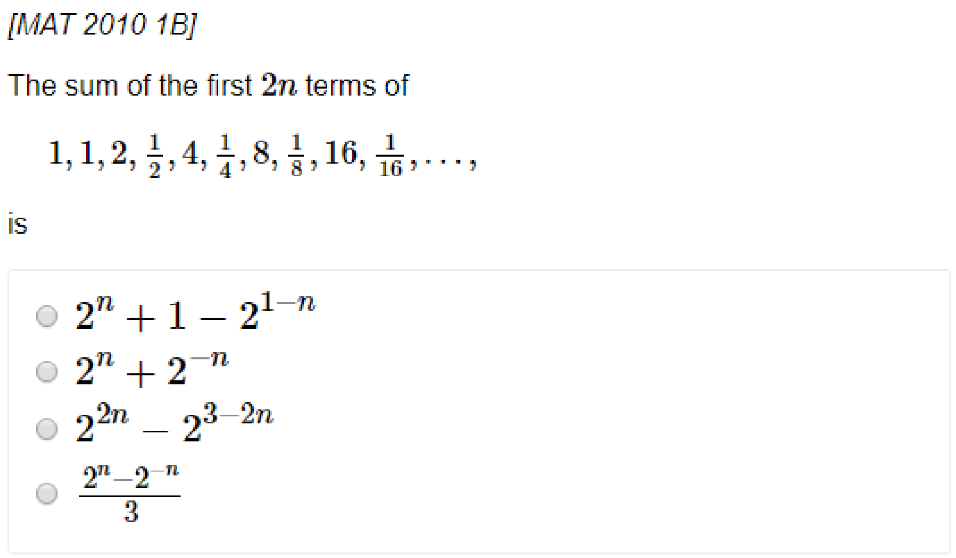
The sum

$$\Sigma_{k=0}^{\infty}\frac{1}{x_k}$$

equals:

Solution: 3

Question 11



[MAT 2010 1B]

The sum of the first $2n$ terms of

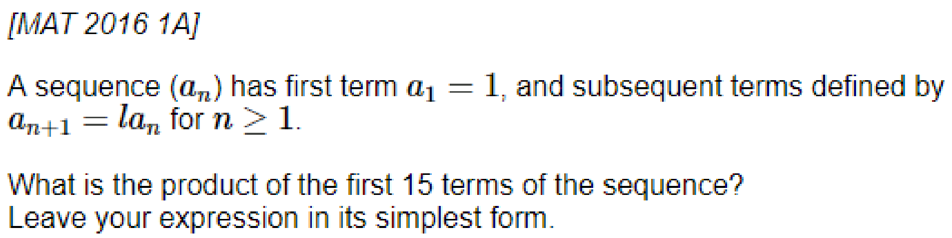
$$1, 1, 2, \tfrac{1}{2}, 4, \tfrac{1}{4}, 8, \tfrac{1}{8}, 16, \tfrac{1}{16}, \ldots,$$

is

○ $2^n + 1 - 2^{1-n}$
○ $2^n + 2^{-n}$
○ $2^{2n} - 2^{3-2n}$
○ $\frac{2^n - 2^{-n}}{3}$

Solution: $2^n + 1 - 2^{1-n}$

Question 12



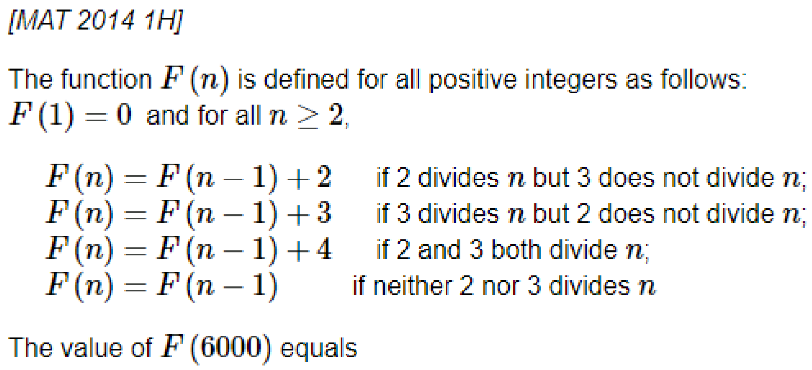
[MAT 2016 1A]

A sequence (a_n) has first term $a_1 = 1$, and subsequent terms defined by $a_{n+1} = la_n$ for $n \geq 1$.

What is the product of the first 15 terms of the sequence?
Leave your expression in its simplest form.

Solution: l^{105}

Question 13



[MAT 2014 1H]

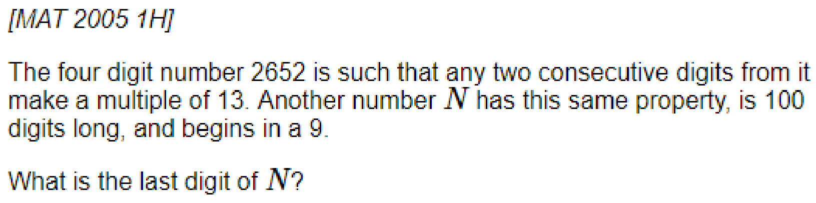
The function $F(n)$ is defined for all positive integers as follows:
$F(1) = 0$ and for all $n \geq 2$,

$F(n) = F(n-1) + 2$ if 2 divides n but 3 does not divide n;
$F(n) = F(n-1) + 3$ if 3 divides n but 2 does not divide n;
$F(n) = F(n-1) + 4$ if 2 and 3 both divide n;
$F(n) = F(n-1)$ if neither 2 nor 3 divides n

The value of $F(6000)$ equals

Solution: 11000

Question 14



[MAT 2005 1H]

The four digit number 2652 is such that any two consecutive digits from it make a multiple of 13. Another number N has this same property, is 100 digits long, and begins in a 9.

What is the last digit of N?

Solution: 9

Question 15 (AEA 2013 Q4)

A sequence of positive integers $a_1, a_2, a_3, \ldots$ has rth term given by

$$a_r = 2^r - 1$$

(a) Write down the first 6 terms of this sequence.

(1)

(b) Verify that $a_{r+1} = 2a_r + 1$

(1)

(c) Find $\sum_{r=1}^{n} a_r$

(3)

(d) Show that $\frac{1}{a_{r+1}} < \frac{1}{2} \times \frac{1}{a_r}$

(1)

(e) Hence show that $1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \ldots < 1 + \frac{1}{3} + \left(\frac{1}{7} + \frac{\frac{1}{2}}{7} + \frac{\frac{1}{4}}{7} + \ldots\right)$

(2)

(f) Show that $\frac{31}{21} < \sum_{r=1}^{\infty} \frac{1}{a_r} < \frac{34}{21}$

(5)

Question	Scheme	Marks	Notes
(a)	$a_1 = 1,\ a_2 = 3,\ a_3 = 7,\ a_4 = 15,\ a_5 = 31,\ a_6 = 63$	B1 (1)	
(b)	Sub: $a_{r+1} = 2^{r+1} - 1;\ \ 2a_r + 1 = \underline{2(2^r - 1) + 1} = 2^{r+1} - 1$	B1cso (1)	Correct demonstration in r
(c)	$\sum a_r = \sum 2^r - \sum 1 = \sum 2^r - n$	B1	For $\sum 1 = n$
	$\sum 2^r = \frac{2\left(2^n - 1\right)}{2-1}$, therefore $\sum a_r = 2(2^n - 1) - n$ (o.e.)	M1 A1 (3)	Use of GP formula Any correct expres' A1 needs – n too.
(d)	$a_{r+1} = 2a_r + 1 \Rightarrow \underline{a_{r+1} > 2a_r} \rightarrow \frac{1}{a_{r+1}} < \frac{1}{2} \times \frac{1}{a_r}$	B1cso (1)	Or equiv in words
(e)	$\frac{1}{a_4} < \frac{\frac{1}{2}}{a_3}$ and $\frac{1}{a_5} < \frac{\frac{1}{2}}{a_4} < \frac{\left(\frac{1}{2}\right)^2}{a_3}$	M1	Use of (d) to get any 2 inequality for 4th and 5th terms
	So: $\sum_{r=1}^{5} \frac{1}{a_r} < 1 + \frac{1}{3} + \frac{1}{7} + \frac{\left(\frac{1}{2}\right)}{7} + \frac{\left(\frac{1}{2}\right)^2 \text{ or } \frac{1}{4}}{7}$	A1cso (2)	All 3 inequalities & no incorrect work
(f)	Lower limit = $1 + \frac{1}{3} + \frac{1}{7} = \frac{31}{21}$	B1cso	
	Identify GP $a = \frac{1}{7},\ \ r = \frac{1}{2}$	M1	Correct r <u>or</u> a
	Use $S_\infty = \frac{\frac{1}{7}}{1 - \frac{1}{2}}\ \left(= \frac{2}{7}\right)$	dM1 A1	Attempt sum $\|r\|<1$ Correct expression or sum
	Upper limit = $1 + \frac{1}{3} + \frac{2}{7} = \frac{34}{21}$	A1cso (5) **(13)**	

Question 16 (AEA 2012 Q3)

The angle θ, $0 < \theta < \frac{\pi}{2}$, satisfies

$$\tan\theta \tan 2\theta = \sum_{r=0}^{\infty} 2\cos^r 2\theta$$

(a) Show that $\tan\theta = 3^p$, where p is a rational number to be found.

(8)

(b) Hence show that $\frac{\pi}{6} < \theta < \frac{\pi}{4}$

(2)

Qu	Scheme	Mark	Notes
(a)	RHS = GP $a = 2, r = \cos 2\theta$ $S_\infty =, \dfrac{2}{1-\cos 2\theta}$	M1,A1	Identify GP and attempt sum to ∞ for M1
	$\cos 2\theta = 1 - 2\sin^2\theta \Rightarrow (\text{RHS}) = \operatorname{cosec}^2\theta$ (Allow $\frac{k}{\sin^2\theta}$)	M1	Use $\cos 2\theta$ to simplify
	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow$ (LHS) $= \dfrac{2\tan^2\theta}{1-\tan^2\theta}$	M1	Use of $\tan 2\theta$ on LHS
	Equating: $\dfrac{2\tan^2\theta}{1-\tan^2\theta} = 1 + \cot^2\theta = \dfrac{1+\tan^2\theta}{\tan^2\theta}$	M1	Equate LHS=RHS and use formula to get eqn in $\tan\theta$ or single trig func.
	so $3\tan^4\theta - 1 = 0$	A1	Correct eqn (either line)
	$\tan^4\theta = \frac{1}{3} \Rightarrow \tan\theta = \left(\frac{1}{3}\right)^{\frac{1}{4}}$	dM1	Solve their equn leading to $\tan\theta = \ldots$ Dep on 4th M
	$\tan\theta = 3^{-\frac{1}{4}}$ or $p = -\dfrac{1}{4}$	A1 (8)	
(b)	$1 > 3^{-\frac{1}{4}} > 3^{-\frac{1}{2}} \Rightarrow \tan\frac{\pi}{4} > \tan\theta > \tan\frac{\pi}{6}$	M1	
	$\Rightarrow \frac{\pi}{4} > \theta > \frac{\pi}{6}$	A1 (2)	cso
		[10]	

Question 17 (AEA 2011 Q3)

A sequence $\{u_n\}$ is given by

$$\begin{aligned} u_1 &= k \\ u_{2n} &= u_{2n-1} \times p \qquad n \geqslant 1 \\ u_{2n+1} &= u_{2n} \times q \qquad n \geqslant 1 \end{aligned}$$

where k, p and q are positive constants with $pq \neq 1$

(a) Write down the first 6 terms of this sequence.

(3)

(b) Show that $\sum_{r=1}^{2n} u_r = \dfrac{k(1+p)\left(1-(pq)^n\right)}{1-pq}$

(6)

In part (c) $[x]$ means the integer part of x, so for example $[2.73] = 2$, $[4] = 4$ and $[0] = 0$

(c) Find $\sum_{r=1}^{\infty} 6 \times \left(\frac{4}{3}\right)^{\left[\frac{r}{2}\right]} \times \left(\frac{3}{5}\right)^{\left[\frac{r-1}{2}\right]}$

(4)

(a)	$k,\ kp,\ kpq;\ \ kp^2q,\ kp^2q^2,\ kp^3q^2$	M1 A2/1/0 (3)	M1 for 1st 3 terms A2/1/0 (-1 eeoo) for next 3
(b)	**[Need one line clearly showing factorisation or split]** Identify: $k+kpq+kp^2q^2\ \dots$ is GP with $a=k,\ r=pq$	M1A1	M1 for splitting into 2 series A1 for 1st *a* and *r*
	Identify: $kp+kp^2q+kp(pq)^2\ \dots$ is GP with $a=kp,\ r=pq$	M1A1	M1 for identifying 2nd GP A1 for 2nd *a* and *r*
	$\mathrm{S}_{2n}=\frac{k\left(1-(pq)^n\right)}{1-pq}+\frac{kp\left(1-(pq)^n\right)}{1-pq}$	M1	Use of S*n* formula twice. One correct ft their *a* & *r*
	$=\frac{k(1+p)\left(1-(pq)^n\right)}{1-pq}$	A1cso (6)	
(c)	$\sum_1^\infty=6+6\times\left(\frac{4}{3}\right)+6\times\left(\frac{4}{3}\right)\times\left(\frac{3}{5}\right)+\dots$ i.e. $k=6,\ p=\frac{4}{3},q=\frac{3}{5}$	B1	Identify link with above and values for *k*, *p* and *q*
	$r=pq=\frac{4}{5}$ ($r<1\ \therefore\ \mathrm{S}_\infty$ formula can be used)	M1	Attempt to find *r*. (S+ for noting $r<1$ etc)
	$\mathrm{S}_\infty=\frac{k(1+p)}{1-pq}=\frac{6\times\frac{7}{3}}{1-\frac{4}{5}},\quad=\frac{210}{3}=\underline{\mathbf{70}}$	A1,A1 (4) **(13)**	A1 for an expression can be in *k*, *p* or *q*. ft their values A1 for 70

Question 18 (AEA 2010 Q2)

The sum of the first p terms of an arithmetic series is q and the sum of the first q terms of the same arithmetic series is p, where p and q are positive integers and $p\neq q$.

Giving simplified answers in terms of p and q, find

(a) the common difference of the terms in this series,

(5)

(b) the first term of the series,

(3)

(c) the sum of the first $(p+q)$ terms of the series.

(3)

(a)	$q=\frac{p}{2}(2a+(p-1)d)$ and $p=\frac{q}{2}(2a+(q-1)d)$	M1 A1	Attempt one sum formula Both correct expressions
	$2\left(\frac{q}{p}-\frac{p}{q}\right)=d(p-1-q+1)$	dM1	Eliminate a. Dep on 1st M1 Must use 2 indep. eqns
		A1	Correct elimination of a
	$d=\frac{2(q^2-p^2)}{pq(p-q)}$; $\underline{d=\frac{-2(p+q)}{pq}}$	A1 (5)	Correct simplified d =
(b)	$2a=\frac{2q}{p}+\frac{(p-1)2(q+p)}{pq}$; $a=\frac{q^2(q-1)-p^2(p-1)}{pq(q-p)}$	M1	Substitute for d in a correct sum formula i.e. eqn in a only
		dM1	Rearrange to a = . Dep M1
	$\frac{q^2+qp+p^2-p-q}{pq}$ or $\frac{q^2+(p-1)(q+p)}{pq}$ or $\frac{p^2+(q-1)(q+p)}{pq}$	A1 (3)	Correct single fraction with denom = pq
(c)	$S_{p+q}=\frac{p+q}{2}\left(\frac{2q}{p}+\frac{(p-1)2(q+p)}{pq}+\frac{-2(p+q)}{pq}(p+q-1)\right)$	M1	Attempt sum formula with $n = (p+q)$ and ft their a and d
	$=\frac{p+q}{2}\left[\frac{2(q^2+qp+p^2-p-q)}{pq}-\frac{2(p+q-1)(p+q)}{pq}\right]$	M1	Attempt to simplify-denominator = pq or $2pq$
	$\frac{p+q}{pq}[-pq]$ = - [p + q]	A1 (3) **[11]**	A1for -(p+q) (S+ for concise simplification/factorising)

Question 19 (AEA 2008 Q1)

The first and second terms of an arithmetic series are 200 and 197.5 respectively.

The sum to n terms of the series is S_n.

Find the largest positive value of S_n.

(Total 5 marks)

① $a=200,\ d=-\frac{5}{2}$ $u_n=0 \Rightarrow 200-\frac{5}{2}(n-1)=0$	Identify a, d and set $u_n=0$	M1
$\Rightarrow$ $n = 81$		A1
[ALT $S_n=\frac{n}{2}(400-\frac{5}{2}(n-1))$  Max at 80.5	S_n and attempt max	M1 A1]
Maximum sum when $n=80$ or 81 $S_{80}=40[400-\frac{5}{2}\times 79]$	Use of S_n with their $n=80$ or their $n=81$	M1 A1
$=20[800-395]$ $=\underline{8100}$		A1 ⑤

Question 20 (AEA 2007 Q5)

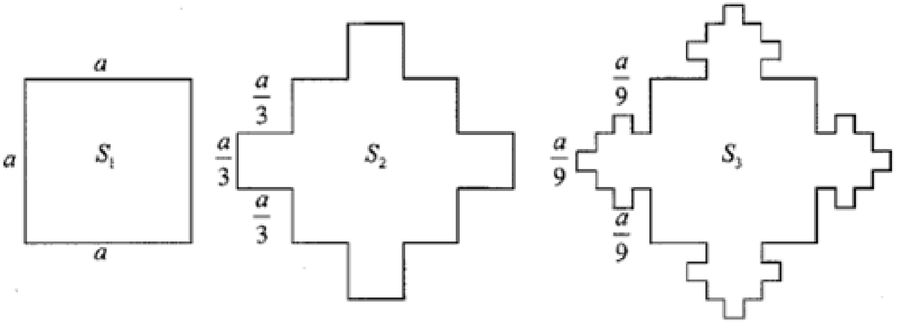


Figure 1

Figure 1 shows part of a sequence $S_1, S_2, S_3, \ldots$, of model snowflakes. The first term S_1 consists of a single square of side a. To obtain S_2, the middle third of each edge is replaced with a new square, of side $\frac{a}{3}$, as shown in Figure 1. Subsequent terms are obtained by replacing the middle third of each external edge of a new square formed in the previous snowflake, by a square $\frac{1}{3}$ of the size, as illustrated by S_3 in Figure 1.

(*a*) Deduce that to form S_4, 36 new squares of side $\frac{a}{27}$ must be added to S_3.

(1)

(*b*) Show that the perimeters of S_2 and S_3 are $\frac{20a}{3}$ and $\frac{28a}{3}$ respectively.

(2)

(*c*) Find the perimeter of S_n.

(4)

(*d*) Describe what happens to the perimeter of S_n as n increases.

(1)

(*e*) Find the areas of S_1, S_2 and S_3.

(2)

(*f*) Find the smallest value of the constant S such that the area of $S_n < S$, for all values of n.

(5)

(a) Each $\left(\frac{a}{9}\right)$ square has 3 sides, there are $4 \times 3 \left(\frac{a}{9}\right)$ squares $\therefore \quad 3 \times 4 \times 3 = 36 \left(\frac{a}{27}\right)$ squares.	Convincing argument or calculation. 3x12 or 9x4 OK 6x6 or 18x2 NOT	B1	(1)
(b) Let P_i = perimeter of S_i $P_1 = 4a, \quad P_2 = 4a + 2 \times \frac{a}{3} \times 4 = 4a + \frac{8a}{3} = \frac{20a}{3}$ $\circledast$	Clear counting method	M1	
$P_3 = P_2 + 2 \times \frac{a}{9} \times 3 \times 4 = P_2 + \frac{8a}{3} = \frac{28a}{3}$ $\circledast$	No incorrect work seen	A1 cso	(2)
(c) $P_1 = 4a$, Common difference $= \frac{8a}{3}$	Identify Arithmetic	M1 A1 (both)	
$\therefore P_n = 4a + (n-1)\frac{8a}{3}$	Use of n^{th} term formula	M1	
$= \underline{\underline{\frac{4a}{3} + \frac{8a}{3}n}}$ or $\underline{\underline{\frac{4a}{3}(2n+1)}}$ or $\underline{\underline{4a + (n-1)\frac{8a}{3}}}$	oe.	A1	(4)
(d) $P_n \to \infty$, as n increases the perimeter $\to \infty$		B1	(1)
Accept perimeter increases.	(continued over)		

Question 1 (STEP I 2011 Q6)

Use the binomial expansion to show that the coefficient of x^r in the expansion of $(1-x)^{-3}$ is $\frac{1}{2}(r+1)(r+2)$.

(i) Show that the coefficient of x^r in the expansion of

$$\frac{1-x+2x^2}{(1-x)^3}$$

is r^2+1 and hence find the sum of the series

$$1+\frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\frac{37}{64}+\frac{50}{128}+\cdots.$$

(ii) Find the sum of the series

$$1+2+\frac{9}{4}+2+\frac{25}{16}+\frac{9}{8}+\frac{49}{64}+\cdots.$$

Solutions: 8 and 12

Question 2 (AEA 2006 Q1)

(*a*) For $|y| < 1$, write down the binomial series expansion of $(1-y)^{-2}$ in ascending powers of y up to and including the term in y^3.

(1)

(*b*) Hence, or otherwise, show that

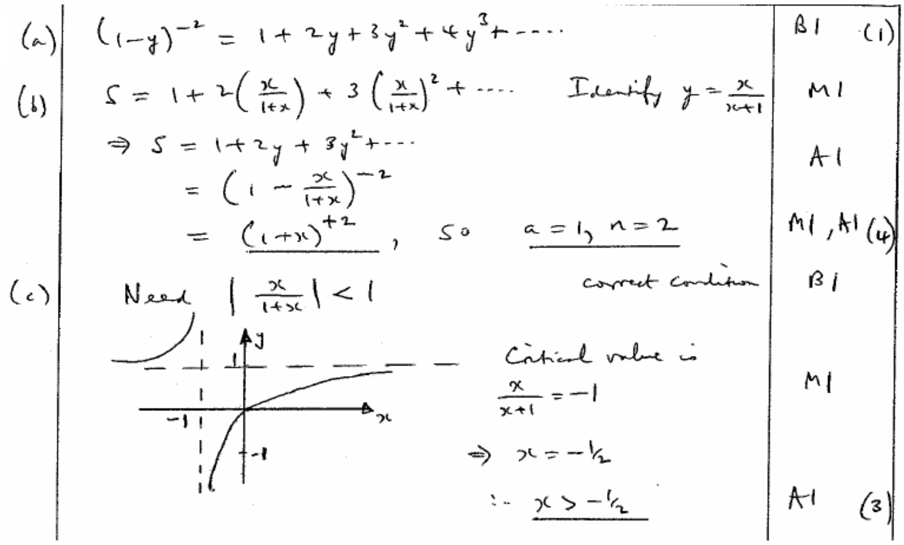
$$1+\frac{2x}{1+x}+\frac{3x^2}{(1+x)^2}+\ldots+\frac{rx^{r-1}}{(1+x)^{r-1}}+\ldots$$

can be written in the form $(a+x)^n$. Write down the values of the integers a and n.

(4)

(*c*) Find the set of values of x for which the series in part (*b*) is convergent.

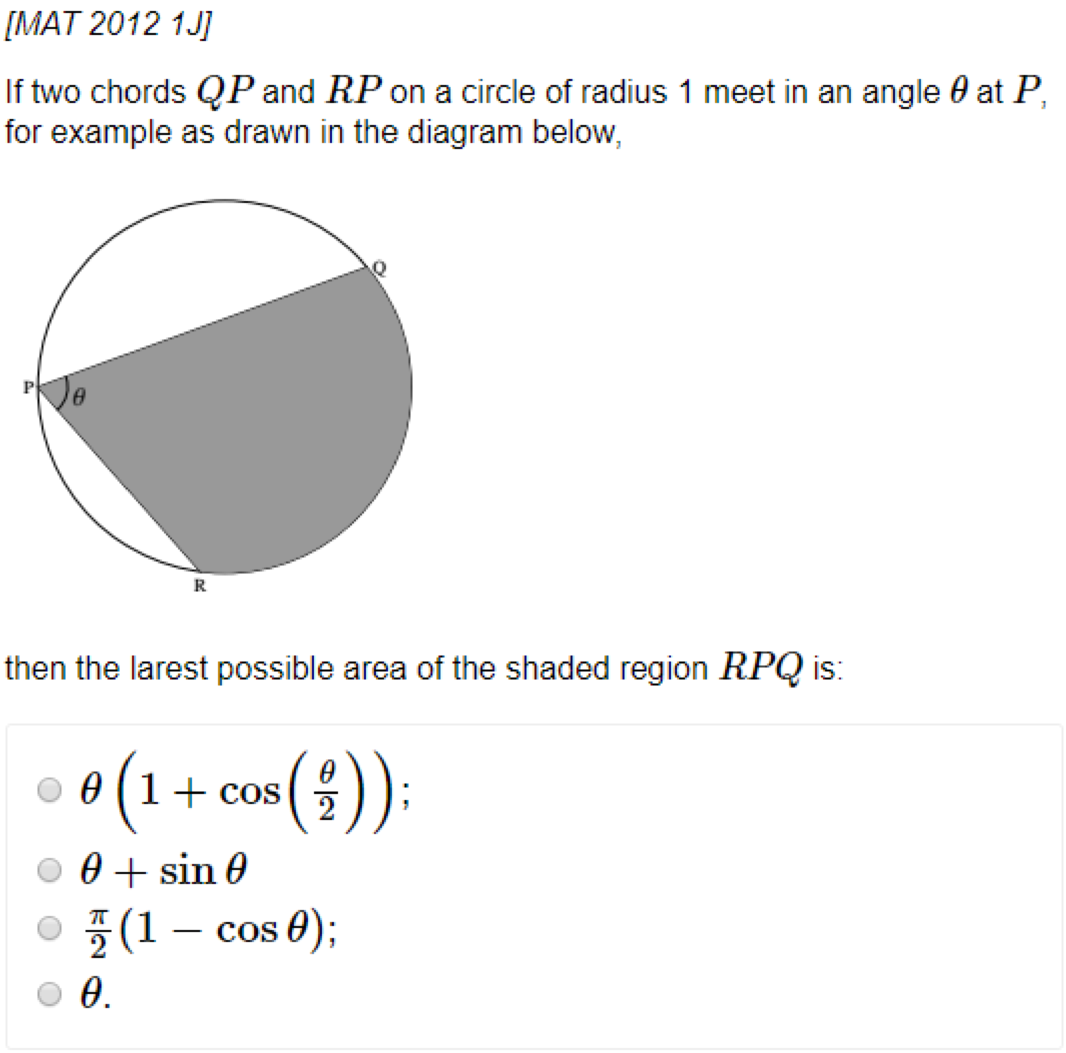
(3)



(a)	$(1-y)^{-2} = 1+2y+3y^2+4y^3+\cdots$	B1 (1)
(b)	$S = 1+2\left(\frac{x}{1+x}\right)+3\left(\frac{x}{1+x}\right)^2+\cdots$ Identify $y=\frac{x}{x+1}$	M1
	$\Rightarrow S = 1+2y+3y^2+\cdots$	A1
	$= \left(1-\frac{x}{1+x}\right)^{-2}$	
	$= \underline{(1+x)^{+2}}$, so $\underline{a=1, n=2}$	M1, A1 (4)
(c)	Need $\left\lvert\frac{x}{1+x}\right\rvert < 1$ correct condition	B1
	Critical value is $\frac{x}{x+1}=-1$	M1
	$\Rightarrow x = -\frac{1}{2}$	
	$\therefore \underline{x > -\frac{1}{2}}$	A1 (3)

2.5 RADIANS (INCLUDING SMALL ANGLE APPROXIMATIONS)

Question 1



[MAT 2012 1J]

If two chords QP and RP on a circle of radius 1 meet in an angle θ at P, for example as drawn in the diagram below,

then the larest possible area of the shaded region RPQ is:

○ $\theta\left(1+\cos\left(\frac{\theta}{2}\right)\right)$;

○ $\theta+\sin\theta$

○ $\frac{\pi}{2}(1-\cos\theta)$;

○ θ.

Solution: $\theta + sin\ \theta$

Question 1 (STEP I 2012 Q6)

A thin circular path with diameter AB is laid on horizontal ground. A vertical flagpole is erected with its base at a point D on the diameter AB. The angles of elevation of the top of the flagpole from A and B are α and β respectively (both are acute). The point C lies on the circular path with DC perpendicular to AB and the angle of elevation of the top of the flagpole from C is ϕ. Show that $\cot\alpha\cot\beta = \cot^2\phi$.

Show that, for any p and q,

$$\cos p\cos q\sin^2\tfrac{1}{2}(p+q) - \sin p\sin q\cos^2\tfrac{1}{2}(p+q) = \tfrac{1}{2}\cos(p+q) - \tfrac{1}{2}\cos(p+q)\cos(p-q).$$

Deduce that, if p and q are positive and $p+q \leqslant \frac{1}{2}\pi$, then

$$\cot p\cot q \geqslant \cot^2\tfrac{1}{2}(p+q)$$

and hence show that $\phi \leqslant \frac{1}{2}(\alpha+\beta)$ when $\alpha+\beta \leqslant \frac{1}{2}\pi$.

Question 2 (AEA 2007 Q1)

(*a*) Write down the binomial expansion of $\dfrac{1}{(1-y)^2}$, $|y| < 1$, in ascending powers of y up to and including the term in y^3.

(1)

(*b*) Hence, or otherwise, show that

$$\frac{1}{4}\operatorname{cosec}^4\left(\frac{\theta}{2}\right) = 1 + 2\cos\theta + 3\cos^2\theta + 4\cos^3\theta + \ldots + (r+1)\cos^r\theta + \ldots$$

and state the values of θ for which this result is not valid.

(4)

Find

(*c*) $$1 + \frac{2}{2} + \frac{3}{2^2} + \frac{4}{2^3} + \ldots + \frac{(r+1)}{2^r} + \ldots,$$

(2)

(*d*) $$1 - \frac{2}{2} + \frac{3}{2^2} - \frac{4}{2^3} + \ldots + (-1)^r\frac{(r+1)}{2^r} + \ldots.$$

(2)

(a) $(1-y)^{-2} = \underline{1 + 2y + 3y^2 + 4y^3}$		B1	(1)
b) Let $y = \cos\theta$, LHS $= \dfrac{1}{(1-\cos\theta)^2}$	Identify $y = \cos\theta$ May be implied	M1	
$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \Rightarrow 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$	$\cos\theta \to \sin\frac{\theta}{2}$	M1	
$\therefore \frac{1}{4}\operatorname{cosec}^4\frac{\theta}{2} = 1 + 2\cos\theta + 3\cos^2\theta + \cdots$ $\circledast$	No incorrect work seen	A1 cso.	
$\lvert y\rvert < 1 \Rightarrow \lvert\cos\theta\rvert < 1$ $\therefore$ <u>not</u> valid for $\theta = n\pi$	Accept $0, \pi, 2\pi, \ldots$	B1	(4)
(c) $y = \frac{1}{2}$, $\Rightarrow$ Sum $= \dfrac{1}{(\frac{1}{2})^2} = \underline{\underline{4}}$	Attempt $y = \frac{1}{2}$ in LHS o.e. for M1	M1 A1	(2)
(d) $y = -\frac{1}{2}$, $\Rightarrow$ Sum $= \dfrac{1}{(3/2)^2} = \underline{\underline{\frac{4}{9}}}$		M1 A1	(2) ⑨

Question 1 (STEP I 2014 Q6)

(i) The sequence of numbers $u_0, u_1, \ldots$ is given by $u_0 = u$ and, for $n \geqslant 0$,

$$u_{n+1} = 4u_n(1 - u_n)\,. \qquad (*)$$

In the case $u = \sin^2\theta$ for some given angle θ, write down and simplify expressions for u_1 and u_2 in terms of θ. Conjecture an expression for u_n and prove your conjecture.

(ii) The sequence of numbers $v_0, v_1, \ldots$ is given by $v_0 = v$ and, for $n \geqslant 0$,

$$v_{n+1} = -pv_n^2 + qv_n + r\,,$$

where p, q and r are given numbers, with $p \neq 0$. Show that a substitution of the form $v_n = \alpha u_n + \beta$, where α and β are suitably chosen, results in the sequence $(*)$ provided that

$$4pr = 8 + 2q - q^2\,.$$

Hence obtain the sequence satisfying $v_0 = 1$ and, for $n \geqslant 0$, $v_{n+1} = -v_n^2 + 2v_n + 2\,.$

Question 2 (STEP I 2010 Q3)

Show that

$$\sin(x + y) - \sin(x - y) = 2\cos x \,\sin y$$

and deduce that

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

Show also that

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A + B)\,\sin\tfrac{1}{2}(A - B)\,.$$

The points P, Q, R and S have coordinates $(a\cos p, b\sin p)$, $(a\cos q, b\sin q)$, $(a\cos r, b\sin r)$ and $(a\cos s, b\sin s)$ respectively, where $0 \leqslant p < q < r < s < 2\pi$, and a and b are positive.

Given that neither of the lines PQ and SR is vertical, show that these lines are parallel if and only if

$$r + s - p - q = 2\pi\,.$$

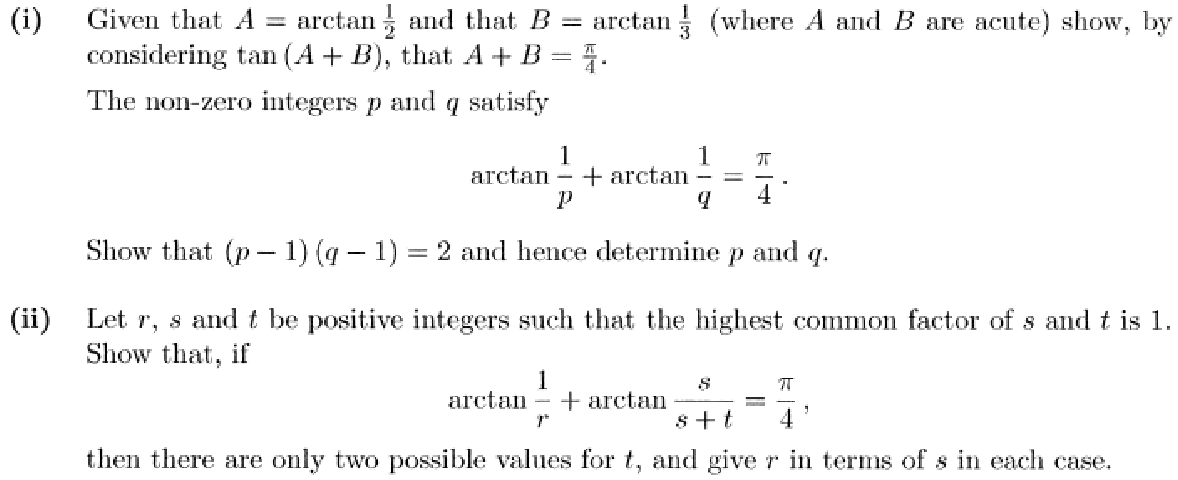
Question 3 (STEP I 2009 Q4)

The sides of a triangle have lengths $p - q$, p and $p + q$, where $p > q > 0$. The largest and smallest angles of the triangle are α and β, respectively. Show by means of the cosine rule that

$$4(1 - \cos\alpha)(1 - \cos\beta) = \cos\alpha + \cos\beta\,.$$

In the case $\alpha = 2\beta$, show that $\cos\beta = \frac{3}{4}$ and hence find the ratio of the lengths of the sides of the triangle.

Question 4 (STEP I 2007 Q2)



(i) Given that $A = \arctan\frac{1}{2}$ and that $B = \arctan\frac{1}{3}$ (where A and B are acute) show, by considering $\tan(A+B)$, that $A + B = \frac{\pi}{4}$.

The non-zero integers p and q satisfy

$$\arctan\frac{1}{p} + \arctan\frac{1}{q} = \frac{\pi}{4}\,.$$

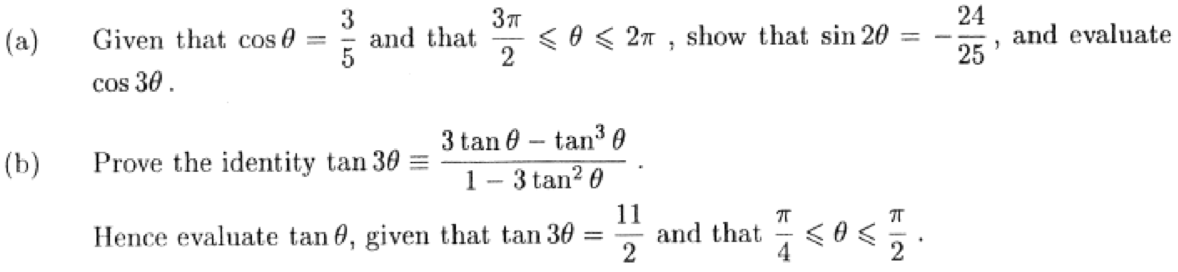
Show that $(p-1)(q-1) = 2$ and hence determine p and q.

(ii) Let r, s and t be positive integers such that the highest common factor of s and t is 1. Show that, if

$$\arctan\frac{1}{r} + \arctan\frac{s}{s+t} = \frac{\pi}{4}\,,$$

then there are only two possible values for t, and give r in terms of s in each case.

Question 5 (STEP I 2005 Q4)



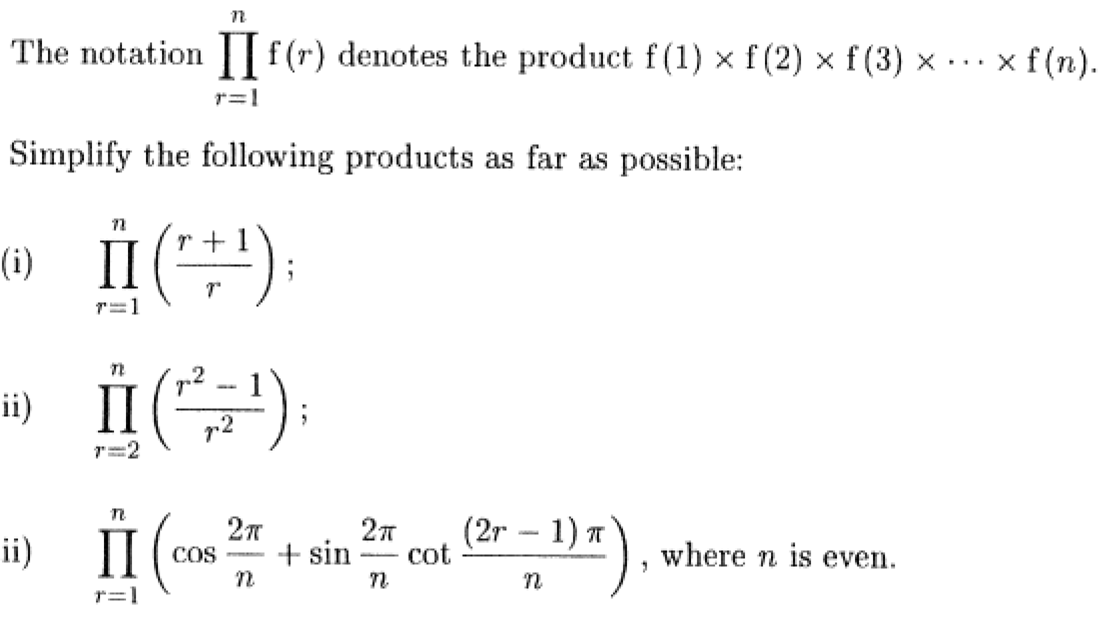
(a) Given that $\cos\theta = \frac{3}{5}$ and that $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\frac{24}{25}$, and evaluate $\cos 3\theta$.

(b) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \frac{11}{2}$ and that $\frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2}$.

Solutions: (a) $-\frac{117}{125}$ (b) $tan\ \theta = 8 + \sqrt{75}$

Question 6 (STEP I 2005 Q7)



The notation $\prod_{r=1}^{n} \mathrm{f}(r)$ denotes the product $\mathrm{f}(1) \times \mathrm{f}(2) \times \mathrm{f}(3) \times \cdots \times \mathrm{f}(n)$.

Simplify the following products as far as possible:

(i) $\displaystyle\prod_{r=1}^{n}\left(\frac{r+1}{r}\right)$;

(ii) $\displaystyle\prod_{r=2}^{n}\left(\frac{r^2-1}{r^2}\right)$;

(iii) $\displaystyle\prod_{r=1}^{n}\left(\cos\frac{2\pi}{n} + \sin\frac{2\pi}{n}\cot\frac{(2r-1)\pi}{n}\right)$, where n is even.

Solutions: (i) $n+1$ (ii) $\frac{n+1}{2n}$ (iii) 1

Question 7 (AEA 2013 Q2)

(a) Use the formula for $\sin(A-B)$ to show that $\sin(90^\circ - x) = \cos x$

(1)

(b) Solve for $0 < \theta < 360^\circ$

$$2\sin(\theta + 17^\circ) = \frac{\cos(\theta + 8^\circ)}{\cos(\theta + 17^\circ)}$$

(7)

Question	**Scheme**	**Marks**	**Notes**
(a)	$\sin(90 - x) = \sin 90 \cos x - \cos 90 \sin x = 1.\cos x - 0.\sin x = \cos x$	B1 (1)	One intermediate line
(b)	$2\sin(\theta+17)\cos(\theta+17) = \cos(\theta+8) \Rightarrow \sin[2(\theta+17)] = \cos(\theta+8)$	M1	Use of sin2A = …
	$2\theta + 34 = 90 - (\theta + 8)$	dM1	Use of (a) – not trigθ
	$3\theta = 82 - 34 = 48$ so **$\underline{\theta = 16}$**	A1	
	$2\theta + 34 = 180 - [90 - (\theta+8)]$ or $2\theta + 34 = [90 - (\theta+8)] + 360$	M1	2nd eqn for θ
	$\theta = 98 - 34$ or **$\underline{\theta = 64}$**	A1	
	$3\theta = 48 + 460$ **$\underline{\theta = 136}$**	A1	
	$\underline{\theta = 256}$	A1 (7)	
NB	$\sin(2\theta+34) - \sin(82 - \theta)$ gives $2\cos[(\theta+116)/2]\sin[(3\theta-48)/2]$	**(8)**	
	Then: $\theta/2 + 58 = 90$ gets M1 and e.g. $3\theta/2 - 24 = 0$ gets M1		

Question 8 (AEA 2011 Q1)

Solve for $0 \leqslant \theta \leqslant 180^\circ$

$$\tan(\theta + 35^\circ) = \cot(\theta - 53^\circ)$$

(Total 4 marks)

Question	Scheme	Marks	Notes
	$\frac{\sin(\theta+35)}{\cos(\theta+35)} = \frac{\cos(\theta-53)}{\sin(\theta-53)}$	M1	Use of correct defns for tan and cot
	$0 = \cos(\theta-53)\cos(\theta+35) - \sin(\theta+35)\sin(\theta-53)$ $0 = \cos(2\theta - 53 + 35)$	M1	Use of cos(A+B) rule to reach single trig function
	$2\theta - 18 = 90, 270$ so **$\underline{x = 54, 144}$**	A1A1 **(4)**	A1 for 54 and A1 for 144
	Use of tan ($A \underline{+} B$) doesn't score until $\tan 2\theta = \tan(90 - -18)$		
ALT	$\tan(\theta+35) = \tan[90 - (\theta - 53)]$	M1	Use of $\cot x = \underline{+}\tan(90 \underline{+} x)$
	$\theta + 35 = 90 - (\theta - 53)$ or $\theta + 35 = 90 - (\theta - 53) + 180$	M1	either

Question 9 (AEA 2009 Q3)

(a) Solve, for $0 \leqslant \theta < 2\pi$,

$$\sin\left(\frac{\pi}{3} - \theta\right) = \frac{1}{\sqrt{3}}\cos\theta \ .$$

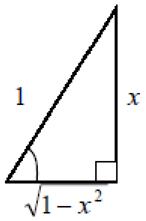
(5)

(b) Find the value of x for which

$$\arcsin(1 - 2x) = \frac{\pi}{3} - \arcsin x, \qquad 0 < x < 0.5$$

[arcsin$x$ is an alternative notation for $\sin^{-1}x$]

(7)

(a)	$\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta \qquad = \frac{1}{\sqrt{3}}\cos\theta$	M1	Use of sin (A - B)
	$\frac{1}{\sqrt{3}}\cos\theta \qquad = \sin\theta$ (o.e)	M1	Use of sin $\frac{\pi}{3}$, cos $\frac{\pi}{3}$ and collect terms
	$\Rightarrow \qquad \tan\theta = \frac{1}{\sqrt{3}}$	A1	$\tan\theta = \frac{1}{\sqrt{3}}$ oe.
	$\theta = \frac{\pi}{6}, \frac{7\pi}{6}$	A1, B1√	
		(5)	
(b)	$\sin\ [\arcsin(1-2x)] = \sin\left[\frac{\pi}{3} - \arcsin x\right]$		
	$\sin[\arcsin(1-2x)] = \sin\frac{\pi}{3}\cos[\arcsin x] - \cos\frac{\pi}{3}\sin(\arcsin x)$	M1	Use of sin $(A \pm B)$
	$1 - 2x \ = \frac{\sqrt{3}}{2}\ \sqrt{1-x^2}\ - \frac{1}{2}\ x$	M1,B1	B1 for cos[arcsinx] $=\sqrt{1-x^2}$
	 $[2 - 3x \ = \sqrt{3}\ \sqrt{1-x^2}\ ]$		
	$4 - 12x + 9x^2 = 3 - 3x^2$	M1	Simplify to quadratic in x
	$12x^2 - 12x + 1$ (=0)	A1	correct 3TQ
	$x = \frac{12 \pm \sqrt{144-48}}{24}$	M1	Attempt to solve if at least one previous M scored in (b)
	$x = \frac{3 \pm \sqrt{6}}{6}$		
	$\because \quad 0 < x < 0.5 \qquad x = \frac{3-\sqrt{6}}{6}$ (o.e.)	A1	Must choose '_'
		(7)	

Question 10 (AEA 2008 Q3)

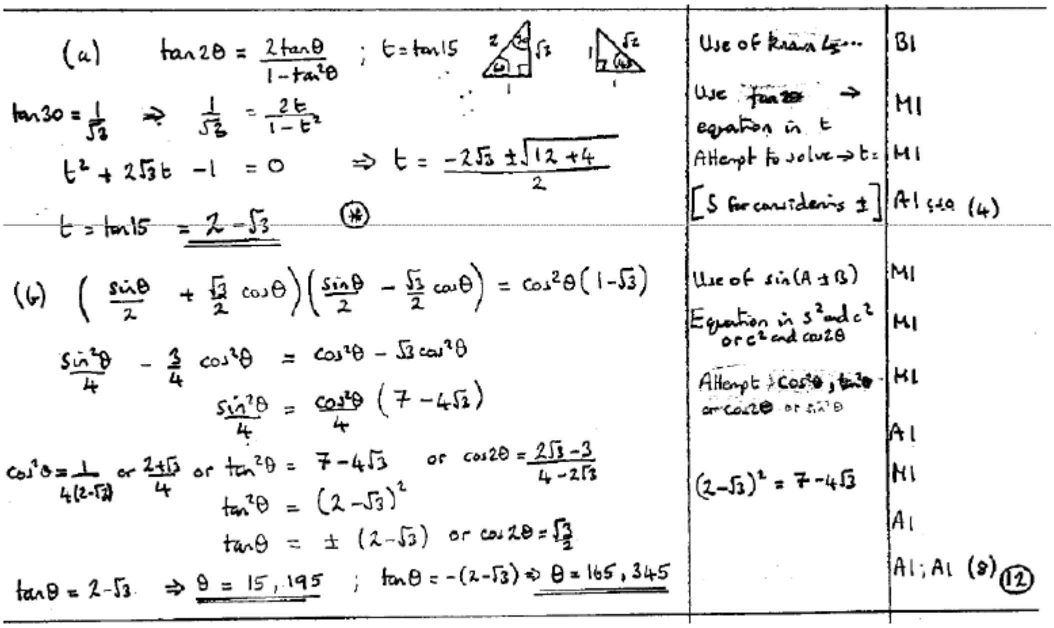
(a) Prove that $\tan 15° = 2 - \sqrt{3}$

(4)

(b) Solve, for $0 \leqslant \theta < 360°$,

$$\sin(\theta + 60°)\sin(\theta - 60°) = (1 - \sqrt{3})\cos^2\theta$$

(8)



(a) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$; $t = \tan 15$	Use of known [illegible]…	B1
$\tan 30 = \frac{1}{\sqrt{3}} \Rightarrow \frac{1}{\sqrt{3}} = \frac{2t}{1-t^2}$	Use tan2θ → equation in t	M1
$t^2 + 2\sqrt{3}t - 1 = 0 \Rightarrow t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$	Attempt to solve → t =	M1
$t = \tan 15 = \underline{\underline{2 - \sqrt{3}}}$ ⊛	[S for considering ±]	A1 cso (4)
(b) $\left(\frac{\sin\theta}{2} + \frac{\sqrt{3}}{2}\cos\theta\right)\left(\frac{\sin\theta}{2} - \frac{\sqrt{3}}{2}\cos\theta\right) = \cos^2\theta(1-\sqrt{3})$	Use of sin(A ± B)	M1
$\frac{\sin^2\theta}{4} - \frac{3}{4}\cos^2\theta = \cos^2\theta - \sqrt{3}\cos^2\theta$	Equation in s^2 and c^2 or c^2 and cos2θ	M1
$\frac{\sin^2\theta}{4} = \frac{\cos^2\theta}{4}(7 - 4\sqrt{3})$	Attempt → cos²θ, tan²θ or cos2θ or sin²θ	M1
$\cos^2\theta = \frac{1}{4(2-\sqrt{3})}$ or $\frac{2+\sqrt{3}}{4}$ or $\tan^2\theta = 7 - 4\sqrt{3}$ or $\cos 2\theta = \frac{2\sqrt{3}-3}{4-2\sqrt{3}}$		A1
$\tan^2\theta = (2-\sqrt{3})^2$	$(2-\sqrt{3})^2 = 7 - 4\sqrt{3}$	M1
$\tan\theta = \pm(2-\sqrt{3})$ or $\cos 2\theta = \frac{\sqrt{3}}{2}$		A1
$\tan\theta = 2-\sqrt{3} \Rightarrow \underline{\theta = 15, 195}$; $\tan\theta = -(2-\sqrt{3}) \Rightarrow \underline{\theta = 165, 345}$		A1; A1 (8) ⑫

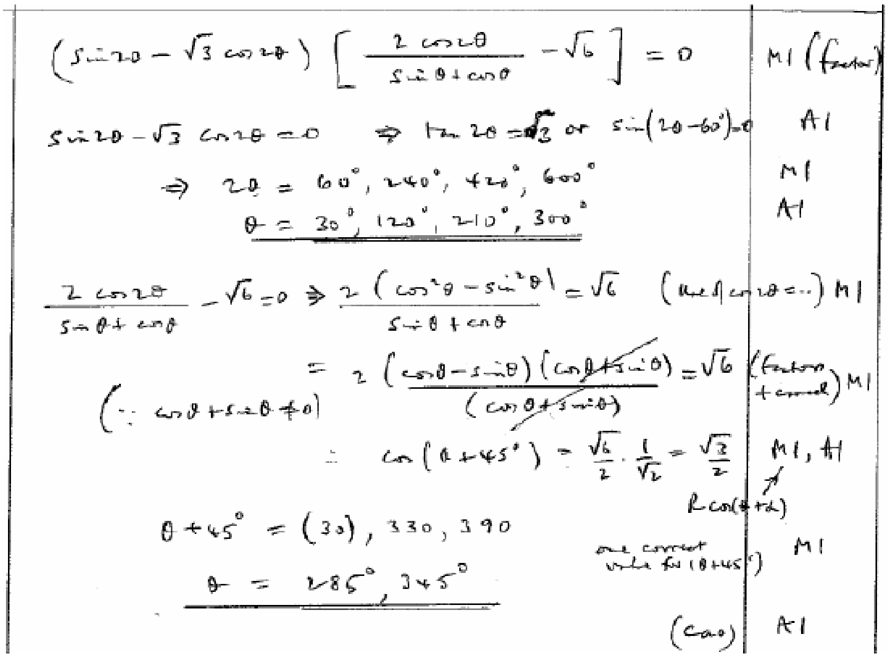
Question 11 (AEA 2006 Q2)

Given that $(\sin\theta + \cos\theta) \neq 0$, find all the solutions of

$$\frac{2\cos 2\theta(\sin 2\theta - \sqrt{3}\cos 2\theta)}{\sin\theta + \cos\theta} = \sqrt{6}(\sin 2\theta - \sqrt{3}\cos 2\theta)$$

for $0 \leq \theta < 360°$.

(10)



$(\sin 2\theta - \sqrt{3}\cos 2\theta)\left[\frac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6}\right] = 0$		M1 (factor)
$\sin 2\theta - \sqrt{3}\cos 2\theta = 0 \Rightarrow \tan 2\theta = \sqrt{3}$ or $\sin(2\theta - 60°) = 0$		A1
$\Rightarrow 2\theta = 60°, 240°, 420°, 600°$		M1
$\underline{\theta = 30°, 120°, 210°, 300°}$		A1
$\frac{2\cos 2\theta}{\sin\theta + \cos\theta} - \sqrt{6} = 0 \Rightarrow \frac{2(\cos^2\theta - \sin^2\theta)}{\sin\theta + \cos\theta} = \sqrt{6}$	(use of cos2θ = ..)	M1
$= 2\frac{(\cos\theta - \sin\theta)\cancel{(\cos\theta + \sin\theta)}}{\cancel{(\cos\theta + \sin\theta)}} = \sqrt{6}$ $(\because \cos\theta + \sin\theta \neq 0)$	(factor + cancel)	M1
$\therefore \cos(\theta + 45°) = \frac{\sqrt{6}}{2}\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{2}$	R cos(θ + α)	M1, A1
$\theta + 45° = (30), 330, 390$	one correct value for (θ + 45°)	M1
$\underline{\theta = 285°, 345°}$	(cao)	A1

Question 12 (AEA 2005 Q2)

Solve, for $0 < \theta < 2\pi$,

$$\sin 2\theta + \cos 2\theta + 1 = \sqrt{6}\cos\theta,$$

giving your answers in terms of π.

(8)

$2\sin\theta\cos\theta + \cos 2\theta + 1 = \sqrt{6}\cos\theta$ (use of $\sin 2\theta =$) M1

$2\sin\theta\cos\theta + 2\cos^2\theta = \sqrt{6}\cos\theta$ ($2\cos^2\theta$ needed) M1

$\cos\theta\left(2\sin\theta + 2\cos\theta - \sqrt{6}\right) = 0$

$\cos\theta = 0 \Rightarrow \theta = \underline{\pi/2, \ 3\pi/2}$ (Factor of $\cos\theta$) M1

(both) A1

or $\sin\theta + \cos\theta = \frac{\sqrt{6}}{2}$

$\sqrt{2}\sin(\theta + \pi/4) = \frac{\sqrt{6}}{2}$ (use of $\sin(\theta + \pi/4)$ or $\cos(\theta - \pi/4)$) M1

$\sin(\theta + \pi/4) = \frac{\sqrt{3}}{2}$ A1

$\theta + \frac{\pi}{4} = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ (2 values) M1

$\underline{\theta = \frac{\pi}{12}, \ \frac{5}{12}\pi}$ (both) A1

(8)

2.8 PARAMETRIC EQUATIONS (PARAMETRIC → CARTESIAN, SKETCHING, POINTS OF INTERSECTION)

Question 1 (STEP 2008 Q7)

The point P has coordinates (x, y) with respect to the origin O. By writing $x = r\cos\theta$ and $y = r\sin\theta$, or otherwise, show that, if the line OP is rotated by 60° clockwise about O, the new y-coordinate of P is $\frac{1}{2}(y - \sqrt{3}\,x)$. What is the new y-coordinate in the case of an anti-clockwise rotation by 60° ?

An equilateral triangle OBC has vertices at O, $(1, 0)$ and $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$, respectively. The point P has coordinates (x, y). The perpendicular distance from P to the line through C and O is h_1; the perpendicular distance from P to the line through O and B is h_2; and the perpendicular distance from P to the line through B and C is h_3.

Show that $h_1 = \frac{1}{2}|y - \sqrt{3}\,x|$ and find expressions for h_2 and h_3.

Show that $h_1 + h_2 + h_3 = \frac{1}{2}\sqrt{3}$ if and only if P lies on or in the triangle OBC.

Solution: (i) $h_2 = |y|$ $\quad h_3 = \frac{1}{2}|y + \sqrt{3}x - \sqrt{3}|$

Question 2 (AEA 2007 Q3)

(*a*) Solve, for $0 \le x < 2\pi$,

$$\cos x + \cos 2x = 0.$$

(5)

(*b*) Find the exact value of x, $x \ge 0$, for which

$$\arccos x + \arccos 2x = \frac{\pi}{2}.$$

(6)

[arccos x is an alternative notation for $\cos^{-1} x$.]

(a) $\cos x + \cos 2x = 0 \Rightarrow \cos x + 2\cos^2 x - 1 = 0$ — $\cos 2x \to \cos x \to$ 3TQ — M1

$(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \frac{1}{2}$ or -1 — A1

$\cos x = \frac{1}{2} \Rightarrow x = \underline{\underline{\frac{\pi}{3}}}, \underline{\underline{\frac{5\pi}{3}}}$ — α, $2\pi - \alpha$ — B1, B1√

$\cos x = -1 \Rightarrow x = \underline{\underline{\pi}}$ — B1

(Condone degrees but SO) — (5)

ALT (a) $\cos x + \cos 2x = 0 \rightarrow (2)\cos\frac{3x}{2}\cos\frac{x}{2} = 0$ — Use of factor formulae — M1

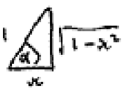
$\Rightarrow \cos\frac{3x}{2} = 0$ or $\cos\frac{x}{2} = 0$ — both — A1

$\cos\frac{3x}{2} = 0 \Rightarrow \frac{3x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots \quad \therefore \quad x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ — B1, B1, B1√

$\left(\cos\frac{x}{2} = 0 \Rightarrow \frac{x}{2} = \frac{\pi}{2}, \dots \quad \therefore \quad x = \pi\right)$ — (5)

(b) Let: $\arccos x = \alpha$, $\arccos 2x = \beta$

$\therefore \quad \cos(\alpha + \beta) = \cos\frac{\pi}{2} \Rightarrow \cos\alpha\cos\beta - \sin\alpha\sin\beta = 0$ — M1

$\cos\alpha = x$  , $\sin\alpha = \sqrt{1-x^2}$ or $\sin\beta = \sqrt{1-4x^2}$ — M1, A1

or $\cos\beta = 2x$ — A1

$\therefore \quad x \times 2x - \sqrt{1-x^2}\sqrt{1-4x^2} = 0$

$4x^4 = (1-x^2)(1-4x^2)$ — M1

$5x^2 = 1$

$\Rightarrow \quad x = \underline{\underline{\frac{1}{\sqrt{5}}}}$ — A1 — (6)

2.9 DIFFERENTIATION (TRIG, EXP/LOG, CHAIN/QUOTIENT/PRODUCT RULE, PARAMETRIC, IMPLICIT, RATES OF CHANGE)

Question 1 (STEP I 2014 Q4)

An accurate clock has an hour hand of length a and a minute hand of length b (where $b > a$), both measured from the pivot at the centre of the clock face. Let x be the distance between the ends of the hands when the angle between the hands is θ, where $0 \leqslant \theta < \pi$.

Show that the rate of increase of x is greatest when $x = (b^2 - a^2)^{\frac{1}{2}}$.

In the case when $b = 2a$ and the clock starts at mid-day (with both hands pointing vertically upwards), show that this occurs for the first time a little less than 11 minutes later.

Question 2 (STEP I 2014 Q5)

(i) Let $\mathrm{f}(x) = (x + 2a)^3 - 27a^2x$, where $a \geqslant 0$. By sketching $\mathrm{f}(x)$, show that $\mathrm{f}(x) \geqslant 0$ for $x \geqslant 0$.

(ii) Use part (i) to find the greatest value of xy^2 in the region of the x-y plane given by $x \geqslant 0$, $y \geqslant 0$ and $x+2y \leqslant 3$. For what values of x and y is this greatest value achieved?

(iii) Use part (i) to show that $(p+q+r)^3 \geqslant 27pqr$ for any non-negative numbers p, q and r. If $(p + q + r)^3 = 27pqr$, what relationship must p, q and r satisfy?

Question 3 (STEP I 2012 Q1)

The line L has equation $y = c - mx$, with $m > 0$ and $c > 0$. It passes through the point $R(a, b)$ and cuts the axes at the points $P(p, 0)$ and $Q(0, q)$, where a, b, p and q are all positive. Find p and q in terms of a, b and m.

As L varies with R remaining fixed, show that the minimum value of the sum of the distances of P and Q from the origin is $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$, and find in a similar form the minimum distance between P and Q. (You may assume that any stationary values of these distances are minima.)

Question 4 (STEP I 2011)

(i) Show that the gradient of the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $b \neq 0$, is $-\dfrac{ay^2}{bx^2}$.

The point (p, q) lies on both the straight line $ax + by = 1$ and the curve $\dfrac{a}{x} + \dfrac{b}{y} = 1$, where $ab \neq 0$. Given that, at this point, the line and the curve have the same gradient, show that $p = \pm q$.

Show further that either $(a - b)^2 = 1$ or $(a + b)^2 = 1$.

(ii) Show that if the straight line $ax + by = 1$, where $ab \neq 0$, is a normal to the curve $\dfrac{a}{x} - \dfrac{b}{y} = 1$, then $a^2 - b^2 = \frac{1}{2}$.

Question 5 (STEP I 2011 Q3)

Prove the identity

$$4\sin\theta\sin(\tfrac{1}{3}\pi-\theta)\sin(\tfrac{1}{3}\pi+\theta)=\sin 3\theta\,. \qquad (*)$$

(i) By differentiating $(*)$, or otherwise, show that

$$\cot\tfrac{1}{9}\pi-\cot\tfrac{2}{9}\pi+\cot\tfrac{4}{9}\pi=\sqrt{3}\,.$$

(ii) By setting $\theta=\frac{1}{6}\pi-\phi$ in $(*)$, or otherwise, obtain a similar identity for $\cos 3\theta$ and deduce that

$$\cot\theta\cot(\tfrac{1}{3}\pi-\theta)\cot(\tfrac{1}{3}\pi+\theta)=\cot 3\theta\,.$$

Show that

$$\operatorname{cosec}\tfrac{1}{9}\pi-\operatorname{cosec}\tfrac{5}{9}\pi+\operatorname{cosec}\tfrac{7}{9}\pi=2\sqrt{3}\,.$$

Question 6 (STEP I 2011 Q4)

The distinct points P and Q, with coordinates $(ap^2,2ap)$ and $(aq^2,2aq)$ respectively, lie on the curve $y^2=4ax$. The tangents to the curve at P and Q meet at the point T. Show that T has coordinates $(apq,a(p+q))$. You may assume that $p\neq 0$ and $q\neq 0$.

The point F has coordinates $(a,0)$ and ϕ is the angle TFP. Show that

$$\cos\phi=\frac{pq+1}{\sqrt{(p^2+1)(q^2+1)}}$$

and deduce that the line FT bisects the angle PFQ.

Question 7 (STEP 1 2011 Q7)

In this question, you may assume that $\ln(1+x) \approx x - \frac{1}{2}x^2$ when $|x|$ is small.

The height of the water in a tank at time t is h. The initial height of the water is H and water flows into the tank at a constant rate. The cross-sectional area of the tank is constant.

(i) Suppose that water leaks out at a rate proportional to the height of the water in the tank, and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = k(\alpha^2 H - h)\,,$$

for some positive constant k. Deduce that the time T taken for the water to reach height αH is given by

$$kT = \ln\left(1 + \frac{1}{\alpha}\right),$$

and that $kT \approx \alpha^{-1}$ for large values of α.

(ii) Suppose that the rate at which water leaks out of the tank is proportional to $\sqrt{h}$ (instead of h), and that when the height reaches $\alpha^2 H$, where α is a constant greater than 1, the height remains constant. Show that the time T' taken for the water to reach height αH is given by

$$cT' = 2\sqrt{H}\left(1 - \sqrt{\alpha} + \alpha\ln\left(1 + \frac{1}{\sqrt{\alpha}}\right)\right)$$

for some positive constant c, and that $cT' \approx \sqrt{H}$ for large values of α.

Question 8 (STEP I 2010 Q2)

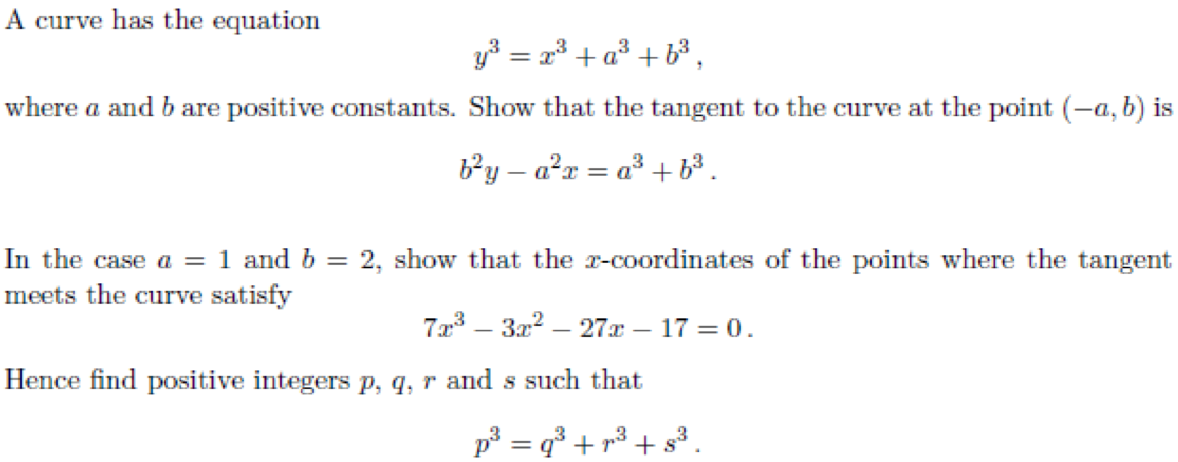
The curve $y = \left(\dfrac{x-a}{x-b}\right)\mathrm{e}^x$, where a and b are constants, has two stationary points. Show that

$$a - b < 0 \quad \text{or} \quad a - b > 4\,.$$

(i) Show that, in the case $a = 0$ and $b = \frac{1}{2}$, there is one stationary point on either side of the curve's vertical asymptote, and sketch the curve.

(ii) Sketch the curve in the case $a = \frac{9}{2}$ and $b = 0$.

Question 9 (STEP I 2009 Q2)



A curve has the equation

$$y^3 = x^3 + a^3 + b^3\,,$$

where a and b are positive constants. Show that the tangent to the curve at the point $(-a, b)$ is

$$b^2y - a^2x = a^3 + b^3\,.$$

In the case $a = 1$ and $b = 2$, show that the x-coordinates of the points where the tangent meets the curve satisfy

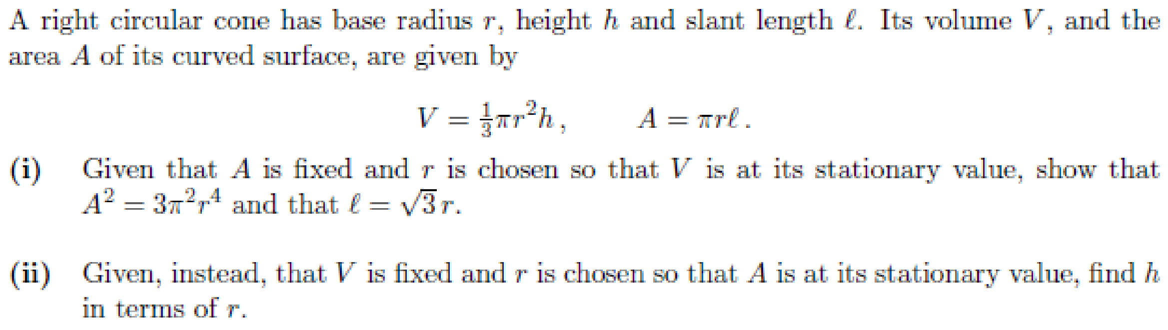
$$7x^3 - 3x^2 - 27x - 17 = 0\,.$$

Hence find positive integers p, q, r and s such that

$$p^3 = q^3 + r^3 + s^3\,.$$

Solution: (ii) $p = 20, q = 17, r = 7, s = 14$

Question 10 (STEP I 2009 Q5)



A right circular cone has base radius r, height h and slant length ℓ. Its volume V, and the area A of its curved surface, are given by

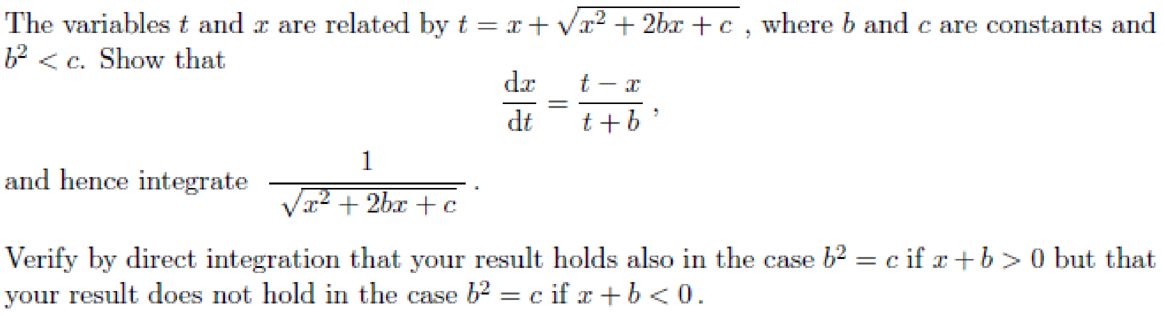
$$V = \tfrac{1}{3}\pi r^2 h\,, \qquad A = \pi r \ell\,.$$

(i) Given that A is fixed and r is chosen so that V is at its stationary value, show that $A^2 = 3\pi^2 r^4$ and that $\ell = \sqrt{3}\,r$.

(ii) Given, instead, that V is fixed and r is chosen so that A is at its stationary value, find h in terms of r.

Solution: (ii) $h = \sqrt{2}r$

Question 11 (STEP I 2008 Q2)



The variables t and x are related by $t = x + \sqrt{x^2 + 2bx + c}$, where b and c are constants and $b^2 < c$. Show that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{t - x}{t + b}\,,$$

and hence integrate $\dfrac{1}{\sqrt{x^2 + 2bx + c}}$.

Verify by direct integration that your result holds also in the case $b^2 = c$ if $x + b > 0$ but that your result does not hold in the case $b^2 = c$ if $x + b < 0$.

Solution: (i) $\ln\left(x + b + \sqrt{x^2 + 2bx + c}\right) + k$

Question 12 (STEP I 2008 Q4)

A function $f(x)$ is said to be *convex* in the interval $a < x < b$ if $f''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $f(x)$ is defined for $0 < x < 2\pi$ by

$$f(x) = e^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $f(x)$ is convex.

(ii) The function $g(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$g(x) = e^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \pi/4$, show that $g(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $g(x)$ is convex.

Solutions: (i) $0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < 2\pi$

(ii) $\frac{\pi}{2} - \alpha < x < \frac{\pi}{2}$

Question 13 (AEA 2011 Q7)

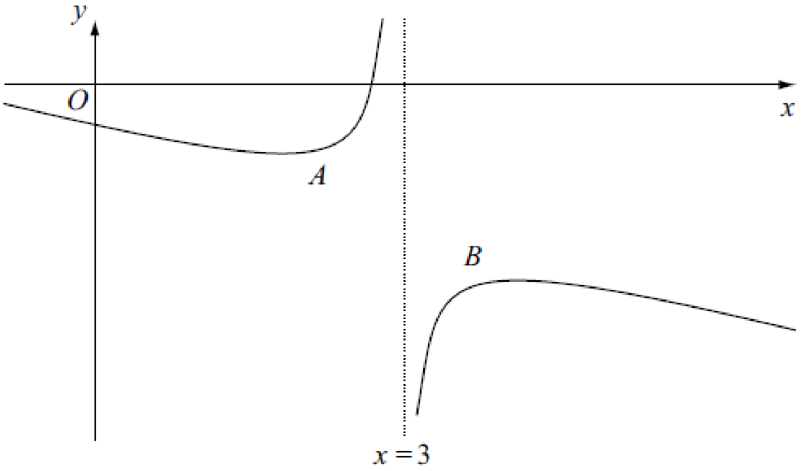


Figure 4

(a) Figure 4 shows a sketch of the curve with equation $y = f(x)$, where

$$f(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R}, \ x \neq 3$$

The curve has a minimum at the point A, with x-coordinate α, and a maximum at the point B, with x-coordinate β.

Find the value of α, the value of β and the y-coordinates of the points A and B.

(5)

(b) The functions g and h are defined as follows

$$\mathrm{g} : x \to x + p \qquad x \in \mathbb{R}$$

$$\mathrm{h} : x \to |x| \qquad x \in \mathbb{R}$$

where p is a constant.

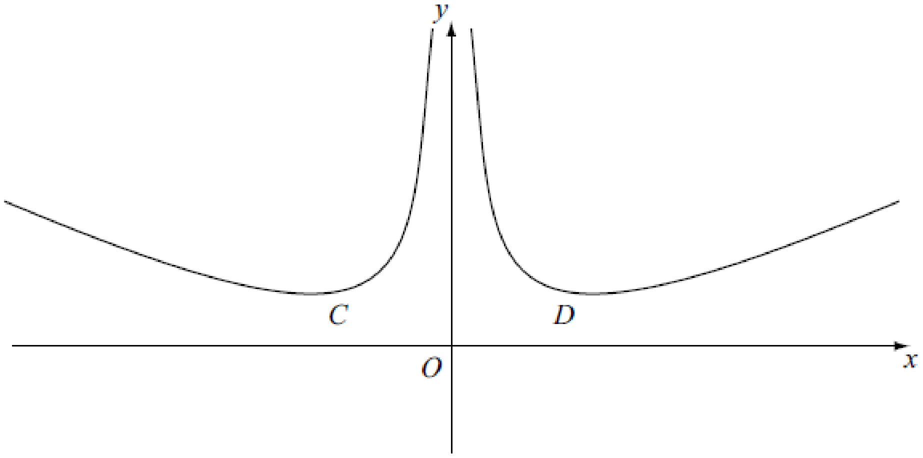


Figure 5

Figure 5 shows a sketch of the curve with equation $y = \mathrm{h}(\mathrm{fg}(x) + q)$, $x \in \mathbb{R}$, $x \neq 0$, where q is a constant. The curve is symmetric about the y-axis and has minimum points at C and D.

(i) Find the value of p and the value of q.

(ii) Write down the coordinates of D.

(5)

(c) The function m is given by

$$\mathrm{m}(x) = \frac{x^2 - 5}{3 - x}, \quad x \in \mathbb{R}, x \leqslant \alpha$$

where α is the x-coordinate of A as found in part (a).

(i) Find m^{-1}

(ii) Write down the domain of m^{-1}

(iii) Find the value of t such that $\mathrm{m}(t) = \mathrm{m}^{-1}(t)$

(10)

(a)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(3-x)2x+\left(x^2-5\right)}{(3-x)^2}$ or $y=-3-x+\frac{4}{3-x}\Rightarrow y'=-1+\frac{4}{(3-x)^2}$	M1 A1	M1 for an attempt to differentiate A1 any correct ver.
	$y'=0\Rightarrow x=1$ or 5	M1	Find stat points
	A is (1, -2) and B is (5, -10)	A1A1 (5)	Full coords
(b) **(i)**	Horizontal translation 3 to left so $p=3$	B1	
	$-2+q=-(q-10)$, so $q=6$	M1A1	M1 for a correct identifiable strategy for b e.g. eqn for q
(ii)	D is (2, 4)	B1B1 (5)	(B1, B1)
(c) **(i)**	$y=\frac{x^2-5}{3-x}\Rightarrow 3y-xy=x^2-5$	M1	Set $y=$ and 1st step
	$3y+5=x^2+yx\Rightarrow\left(x+\frac{y}{2}\right)^2=3y+5+\frac{y^2}{4}$	M1	Isolate x's or set up as 3TQ and attempt to solve for x
	$x+\frac{y}{2}=\pm\frac{\sqrt{y^2+12y+20}}{2}$ o.e. (Accept +, − or ±)	A1	[S+ for reason for choosing −]
	$x=\frac{-y-\sqrt{y^2+12y+20}}{2}$ $\left[\text{so } \mathrm{m}^{-1}(x)=\frac{-x-\sqrt{x^2+12x+20}}{2}\right]$	A1 (4)	Must choose −
(ii)	Domain is range of m(x) i.e. $(x\in ¡\ ,)\,x\geq-2$	B1 (1)	
(iii)	If $\mathrm{m}(t)=\mathrm{m}^{-1}(t)$ then m(x) intersects with $y=x$	M1	Suitable strategy leading to an eqn for t. ft their m^{-1}
	$\frac{t^2-5}{3-t}=t$	A1	A correct quadratic equation
	$2t^2-3t-5(=0)$	M1	Solving correct 3TQ
	$(2t-5)(t+1)=0$	A1	correct factors (A1)
	$t=-1$ (or 2.5)	A1 (5)	(-1 only)
	Can't be 2.5 since not in domain for m(x)	**(20)**	[S+ for reason]

Question 14 (AEA 2010 Q3)

The curve C has equation

$$x^2+y^2+fxy=g^2,$$

where f and g are constants and $g\neq 0$.

(a) Find an expression in terms of α, β and f for the gradient of C at the point (α, β).

(4)

Given that $f<2$ and $f\neq -2$ and that the gradient of C at the point (α, β) is 1,

(b) show that $\alpha=-\beta=\dfrac{\pm g}{\sqrt{(2-f)}}$.

(4)

Given that $f=-2$,

(c) sketch C.

(3)

(a)	$2x+2yy'+fy+fxy'=0$	M1	Correct attempt to diff'n y^2 or xy
		A1	All fully correct and = 0
	$\therefore y'=\dfrac{2x+fy}{-[2y+fx]}$	dM1	Isolate y' Dep on 1st M1
	At (α,β) gradient, $m=\dfrac{2\alpha+f\beta}{-[2\beta+f\alpha]}$ (o.e.)	A1 (4)	Sub α and β
(b)	$m=1$ gives: $2\alpha+f\beta=-2\beta-f\alpha$	M1	Sub m = 1 and form linear equation in α and β.
	$\therefore(\alpha+\beta)(f+2)=0\Rightarrow\alpha=-\beta\quad(\text{or } f=-2)$ (*)	A1cso	(S+ for using $f\neq-2$)
	From curve: $\alpha^2+\alpha^2-f\alpha^2-g^2=0$ (o.e.)	M1	Sub $(\alpha=-\beta)$into equation of curve
	$\therefore\alpha^2(2-f)=g^2\Rightarrow\alpha^2=\dfrac{g^2}{2-f}$ and so $\alpha(\text{or } \beta)=\dfrac{\pm g}{\sqrt{2-f}}$ (*)	A1cso (4)	Simplify to answer. (S+ for considering $f<2$)
(c)	$(x-y)^2=g^2$ or $x-y=\pm g$	M1	Attempt to complete the square, allow ± Or shows m = 1
	Line $y=x+g$ sketched	A1	Sketches should show y intercept or eq'n at least.
	Line $y=x-g$ sketched	A1 (3) **[11]**	

Question 15 (AEA 2010 Q6)

(a) Given that $x^4 + y^4 = 1$, prove that $x^2 + y^2$ is a maximum when $x = \pm y$, and find the maximum and minimum values of $x^2 + y^2$.

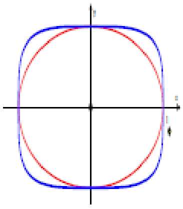
(7)

(b) On the same diagram, sketch the curves C_1 and C_2 with equations $x^4 + y^4 = 1$ and $x^2 + y^2 = 1$ respectively.

(2)

(c) Write down the equation of the circle C_3, centre the origin, which touches the curve C_1 at the points where $x = \pm y$.

(1)

(a)	$A = x^2 + y^2 = x^2 + (1-x^4)^{\frac{1}{2}}$	B1	A as function of x only
	$\therefore \frac{dA}{dx} = 2x - (2x^3)(1-x^4)^{-\frac{1}{2}}$	M1	For some correct diff'n. More than just $2x$
	$\frac{dA}{dx} = 0, \quad x = 0$ or $x^2 = (1-x^4)^{\frac{1}{2}}$	A1 B1	For $x^2 = (1-x^4)^{\frac{1}{2}}$ For $x = 0[\Rightarrow$ by min = 1]
	i.e. $x^2 = y^2 \Rightarrow x = \pm y$; and $x^4 = y^4 = \frac{1}{2}$, so $x^2 + y^2 = \sqrt{2}$	M1; B1	M1 for reaching $y = \underline{+}x$ B1 for max = $\sqrt{2}$
	So minimum is 1 [and maximum is $\sqrt{2}$]	B1 (7)	For min = 1
(b)		B1	Circle, centre (0,0) r =1
		B1	Other curve
(c)	$x^2 + y^2 = \sqrt{2}$	B1 (3) **[10]**	(S+ for some explanation
ALT(a)	Let $x = r\cos\theta$ and $y = r\sin\theta$ then $r^4(\cos^4\theta + \sin^4\theta) = 1$	B1	
	$r^4 = \frac{1}{\cos^4\theta + \sin^4\theta} = \frac{1}{1 - \frac{1}{2}\sin^2 2\theta}$; So $1 < r^2 < 2$	M1A1; B1B1	
	Max value when $\theta = \frac{\pi}{4}$ so $x = y$	M1A1	
OR	$A^2 = (x^2+y^2)^2 = 1 + 2x^2y^2 = 1 + 2x^2\sqrt{(1-x^4)}$	1^{st} B1	Then differentiate as before
OR	$A^2 - 1 = 2x^2y^2 \rightarrow (A^2-1)^2 = 4x^4(1-x^4)$; $= 4(\frac{1}{4} - (\frac{1}{2} - x^4)^2)$	B1:M1A1	By completing the square

Question 16 (AEA 2009 Q2)

The curve C has equation $y = x^{\sin x}$, $\quad x > 0$.

(a) Find the equation of the tangent to C at the point where $x = \frac{\pi}{2}$.

(6)

(b) Prove that this tangent touches C at infinitely many points.

(3)

(a)	$y = x^{\sin x}$ so when $x = \frac{\pi}{2} \Rightarrow y = \frac{\pi}{2}^1 \qquad = \frac{\pi}{2}$	B1	
	$\ln y = \sin x \ln x$	M1	Use of logs (o.e)
	$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{\sin x}{x}$	M1	Use of product rule
		A1	
	$\left[\frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)\right]$		Some correct sub in their y'
	at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ gradient $= \frac{\pi}{2}\left(0 + \frac{1}{\pi/2}\right) = 1$	M1	$\left.\frac{dy}{dx}\right\vert_{x=\pi/2}$
	$\therefore$ Equation of tangent is $\underline{y = x}$	A1 cso (6)	
(b)	If it touches again then $y = x \Rightarrow \sin x = 1$	M1	Method $\longrightarrow \sin x = 1$
	$\Rightarrow x = \frac{\pi}{2} + 2n\pi$	A1	May be listed…
	Gradient at $\left(\frac{\pi}{2} + 2n\pi\right)$ is $\left(\frac{\pi}{2} + 2n\pi\right)\left[0 + \frac{1}{\frac{\pi}{2} + 2n\pi}\right] = 1$	A1 (3)	Check points satisfy $m = 1$ plus comment
	$\therefore$ at points $\left(\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$ $y = x$ is a tangent.	**[9]**	

Question 17 (AEA 2009 Q4)

(a) The function $\mathrm{f}(x)$ has $\mathrm{f}'(x) = \dfrac{\mathrm{u}(x)}{\mathrm{v}(x)}$. Given that $\mathrm{f}'(k) = 0$,

show that $\mathrm{f}''(k) = \dfrac{\mathrm{u}'(k)}{\mathrm{v}(k)}$.

(3)

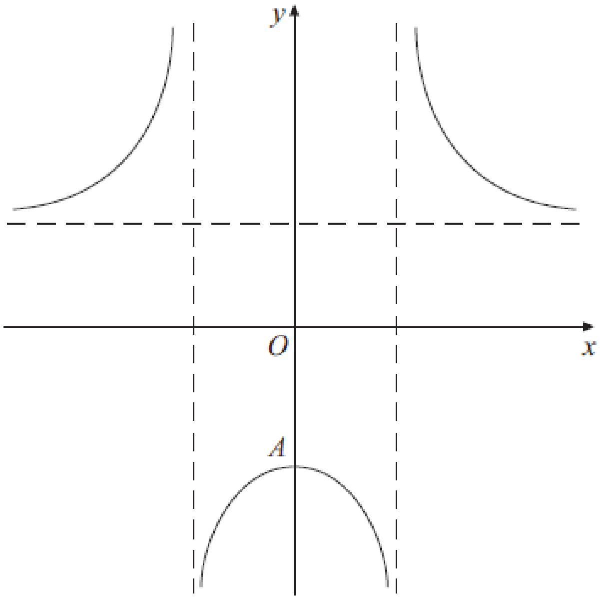


Figure 1

(b) The curve C with equation

$$y = \frac{2x^2 + 3}{x^2 - 1}$$

crosses the y-axis at the point A. Figure 1 shows a sketch of C together with its 3 asymptotes.

(i) Find the coordinates of the point A.

(1)

(ii) Find the equations of the asymptotes of C.

(2)

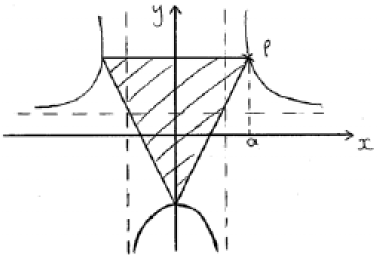
The point $P(a, b)$, $a > 0$ and $b > 0$, lies on C. The point Q also lies on C with PQ parallel to the x-axis and $AP = AQ$.

(iii) Show that the area of triangle PAQ is given by $\dfrac{5a^3}{a^2 - 1}$.

(2)

(iv) Find, as a varies, the minimum area of triangle PAQ, giving your answer in its simplest form.

(6)

(a)	$f''(x)=\frac{vu^1-uv^1}{v^2}$	M1	Use of Quotient rule
	$f'(k)=0 \Rightarrow u(k)=0 \qquad \therefore \quad f''(k)=\frac{vu^1-0}{v^2}$	M1	Sub $u(k)=0$
	$\therefore \quad f''(k)=\frac{u^1(k)}{v(k)}$ (*) (accept $\frac{u^1}{v}$)	A1 cso (3)	Insist on k not x
(b) (i)	A (0, -3)	B1 (1)	Accept $y=-3$
(ii)	Asymptotes $x=1, x=-1$	B1	Both
	and $y=2$	B1 (2)	
	 Area, $T=\frac{1}{2}\times 2a\times(b+3)$	M1	Any correct exp. for T in terms of a and b or complete 2nd line
	$T=a\left[\frac{2a^2+3}{a^2-1}+3\right]=\frac{5a^3}{a^2-1}$ (*)	A1 cso (2)	
(iv)	$\frac{dT}{da}=\frac{(a^2-1)15a^2-5a^3 2a}{(a^2-1)^2}$	M1	Use of quotient rule to find $\frac{dT}{da}$
	$=\frac{5a^2(3a^2-3-2a^2)}{(a^2-1)^2} \quad = \quad \frac{5a^2(a^2-3)}{(a^2-1)^2}$	M1	Solving $\frac{dy}{dx}=0$ --> $a=\ldots$ or $a^2=\ldots$
	$\frac{dT}{da}=0 \quad \Rightarrow a^2=3$ or $a=\sqrt{3}$ (or $a=0$ but $a>0$)	A1 (S+)	Condone $a=\pm\sqrt{3}$
	$\frac{dT}{da}=\frac{5a^4-15a^2}{(a^2-1)^2}$ compare $\frac{u}{v}$ $\therefore \left.\frac{d^2T}{da^2}\right\|_{a=\sqrt{3}}=\left.\frac{20a^3-30a}{(a^2-1)^2}\right\|_{a=\sqrt{3}}$	M1	Full method e.g. $T''(\sqrt{3})$ attempted
	T" $(\sqrt{3})=\frac{60\sqrt{3}-30\sqrt{3}}{4}=\left(\frac{15\sqrt{3}}{2}\right)>0 \quad \therefore$ min	A1	Full accuracy + comment
	$\therefore$ Minimum area $=\frac{5\sqrt{3}\times 3}{3-1} \quad = \quad \frac{15\sqrt{3}}{2}$	B1 (6)	Must come from $T(\sqrt{3})$ not $T''(\sqrt{3})$
		[14]	
	N.B $\frac{d^2T}{da^2}=\frac{10a(a^2+3)}{(a^2-1)^3}$ or $\frac{10a(a^4+2a^2-3)}{(a^2-1)^4}$		Suggest S1 > 12 S2 for S+ and 13 or 14.
	ALT for (iv) Attempt $\frac{d^2T}{da^2}=\ \ldots$	M1	No value of a needed.
	Correct $\frac{d^2T}{da^2}$ and comment.	A1	Fully correct and full comment.

Question 18 (AEA 2007 Q6)

Figure 2

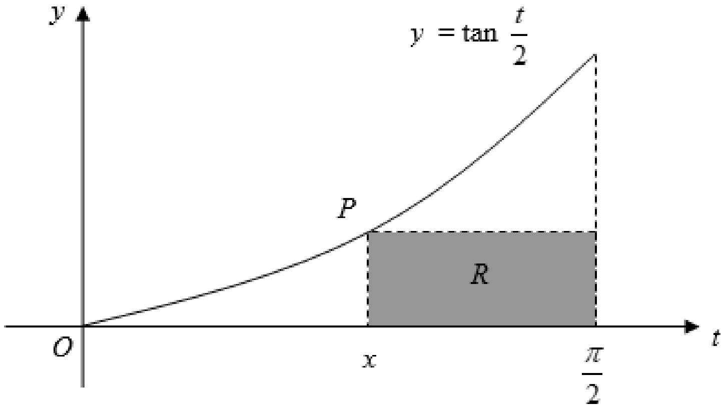


Figure 2 shows a sketch of the curve C with equation $y = \tan \frac{t}{2}$, $0 \leq t \leq \frac{\pi}{2}$.

The point P on C has coordinates $\left(x, \tan \frac{x}{2}\right)$.

The vertices of rectangle R are at $(x, 0)$, $\left(\frac{x}{2}, 0\right)$, $\left(\frac{x}{2}, \tan \frac{x}{2}\right)$ and $\left(x, \tan \frac{x}{2}\right)$ as shown in Figure 2.

(*a*) Find an expression, in terms of x, for the area A of R. **(1)**

(*b*) Show that $\frac{dA}{dx} = \frac{1}{4}(\pi - 2x - 2 \sin x) \sec^2 \frac{x}{2}$. **(4)**

(*c*) Prove that the maximum value of A occurs when $\frac{\pi}{4} < x < \frac{\pi}{3}$. **(7)**

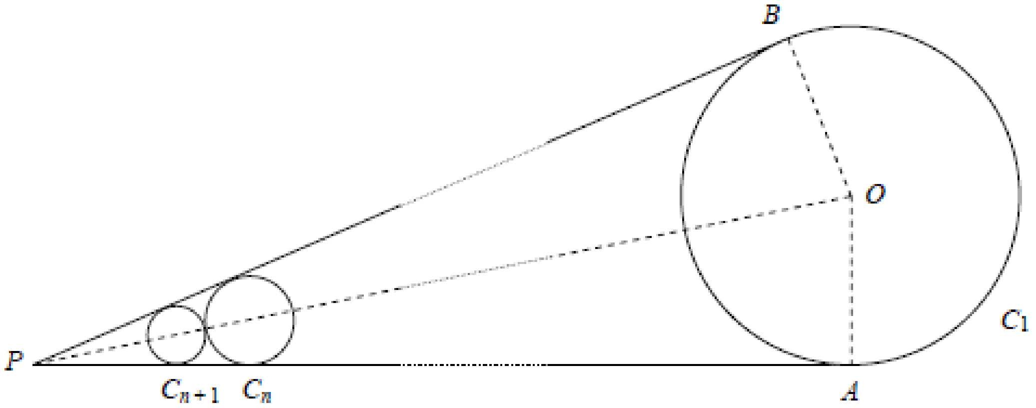
(*d*) Prove that $\tan \frac{\pi}{8} = \sqrt{2} - 1$. **(3)**

(*e*) Show that the maximum value of $A > \frac{\pi}{4}(\sqrt{2} - 1)$. **(2)**

(a) $A = \tan\left(\frac{x}{2}\right)\left(\frac{\pi}{2} - x\right)$	o.e.	B1	(1)
(b) $\frac{dA}{dx} = \frac{1}{2}\sec^2\frac{x}{2}\left(\frac{\pi}{2} - x\right) - \tan\frac{x}{2}$	M for use of product rule	M1 A1	
N.B. $2\sin x\sec^2\frac{x}{2} = 4\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}\sec^2\frac{x}{2} = 4\tan\frac{x}{2}$	Use of $\sin 2A = \cdots$ o.e.	M1 (Must see this)	
$\therefore \frac{dA}{dx} = \frac{1}{4}\sec^2\frac{x}{2}\left(\pi - 2x - 2\sin x\right)$ (*)	No incorrect working seen	A1 cso	(4)
(c) $A'\left(\frac{\pi}{4}\right) = +ve\left(\pi - \frac{\pi}{2} - \frac{2}{\sqrt{2}}\right), = +ve\left(\frac{\pi}{2} - \sqrt{2}\right) > 0$	$A'\left(\frac{\pi}{4}\right)$ attempted > 0	M1 A1	[5 marks for reasons]
$A'\left(\frac{\pi}{3}\right) = +ve\left(\pi - \frac{2\pi}{3} - \sqrt{3}\right), = ve\left(\frac{\pi}{3} - \sqrt{3}\right) < 0$	$A'\left(\frac{\pi}{3}\right)$ attempted < 0	M1 A1	
(Change of sign) ⇒ stationary point for $\frac{\pi}{4} < x < \frac{\pi}{3}$		M1 A1	
∵ gradient moves from >0 to <0 /‾\ ⇒ max	convincing argument	A1	(7)
(d) Let $t = \tan\frac{\pi}{8}$, $\left(\tan\frac{\pi}{4} =\right) 1 = \frac{2t}{1-t^2}$	attempt eqn in t	M1	
$t^2 + 2t - 1 = 0 \Rightarrow t = \frac{-2 \pm \sqrt{4+4}}{2}$	attempt to solve 3TQ	M1	
$\left(\because \frac{\pi}{8} \text{ is acute } \therefore t > 0\right) \Rightarrow \underline{t = \sqrt{2} - 1}$ (*)	(for 5 marks)	A1 cso	(3)
(e) $A_{max} > A\left(\frac{\pi}{4}\right)$	+Attempt @ $A\left(\frac{\pi}{4}\right)$	M1	
$\therefore A_{max} > \tan\frac{\pi}{8}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$			
i.e. $A_{max} > (\sqrt{2}-1)\frac{\pi}{4}$	No incorrect working seen	A1 cso.	(2)
			⑰

Question 19 (AEA 2006 Q7)

Figure 2



The circle C_1 has centre O and radius R. The tangents AP and BP to C_1 meet at the point P and angle $APB = 2\alpha$, $0 < \alpha < \frac{\pi}{2}$. A sequence of circles $C_1, C_2, \ldots, C_n, \ldots$ is drawn so that each new circle C_{n+1} touches each of C_n, AP and BP for $n = 1, 2, 3, \ldots$ as shown in Figure 2. The centre of each circle lies on the line OP.

(*a*) Show that the radii of the circles form a geometric sequence with common ratio

$$\frac{1-\sin\alpha}{1+\sin\alpha}.$$

(5)

(*b*) Find, in terms of R and α, the total area enclosed by all the circles, simplifying your answer.

(3)

The area inside the quadrilateral $PAOB$, not enclosed by part of C_1 or any of the other circles, is S.

(*c*) Show that

$$S = R^2\left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right).$$

(5)

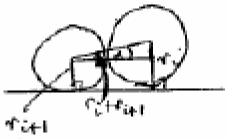
(*d*) Show that, as α varies,

$$\frac{\mathrm{d}S}{\mathrm{d}\alpha} = R^2\cot^2\alpha\left(\frac{\pi}{4}\cos\alpha - 1\right).$$

(4)

(*e*) Find, in terms of R, the least value of S for $\frac{\pi}{6} \leq \alpha \leq \frac{\pi}{4}$.

(3)

(a)	 Appropriate figure	M1
	$\Rightarrow \sin\alpha = \dfrac{r_i - r_{i+1}}{r_i + r_{i+1}}$ (Acp. for $\sin\alpha$)	M1, A1
	$\therefore (r_i + r_{i+1})\sin\alpha = r_i - r_{i+1}$	
	$\therefore r_{i+1}(1+\sin\alpha) = r_i(1-\sin\alpha)$	$\left(\frac{r_{i+1}}{r_i}\right)$ M1
	$\therefore$ ratio of radii $= \underline{\dfrac{1-\sin\alpha}{1+\sin\alpha}}$ ⊛ $(=r)$	A1 cso (5)
(b)	Total area $= \pi R^2 + \pi r_2^2 + \pi r_3^2 + \ldots$	
	$= \pi R^2(1 + r^2 + r^4 + \ldots)$ (correct "r")	B1
	$= \dfrac{\pi R^2}{1-r^2} = \pi R^2 \dfrac{1}{1-\left(\frac{1-\sin\alpha}{1+\sin\alpha}\right)^2}$	M1
	$= \dfrac{\pi R^2(1+\sin\alpha)^2}{(1+\sin\alpha)^2-(1-\sin\alpha)^2} = \underline{\underline{\dfrac{\pi R^2(1+\sin\alpha)^2}{4\sin\alpha}}}$	A1 (3)
(c)	Required area $= 2 \times$ Area ΔPOA + Area major sector AOB $-$ Area found in (b).	M1
	Area $\Delta POA = \frac{1}{2}R.(R\cot\alpha)$	B1
	$\angle POA = \pi/_2 - \alpha$ $\therefore$ angle of major sector $AOB = \pi + 2\alpha$	M1
	$\therefore$ Area sector $AOB = \frac{1}{2}R^2(\pi + 2\alpha)$	A1
	$\therefore$ Required area $= R^2\left(\cot\alpha + \frac{\pi}{2} + \alpha - \frac{\pi}{4}\left(\frac{1+2\sin\alpha+\sin^2\alpha}{\sin\alpha}\right)\right)$	
	$= \underline{R^2\left(\alpha + \cot\alpha - \frac{\pi}{4}\operatorname{cosec}\alpha - \frac{\pi}{4}\sin\alpha\right)}$	⊛ A1 cso (5)

Question 20 (AEA 2005 Q4)

A rectangle *ABCD* is drawn so that *A* and *B* lie on the *x*-axis, and *C* and *D* lie on the curve with equation $y = \cos x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. The point *A* has coordinates $(p, 0)$, where $0 < p < \frac{\pi}{2}$.

(*a*) Find an expression, in terms of *p*, for the area of this rectangle.

(2)

The maximum area of *ABCD* is *S* and occurs when $p = \alpha$. Show that

(*b*) $\frac{\pi}{4} < \alpha < 1$,

(6)

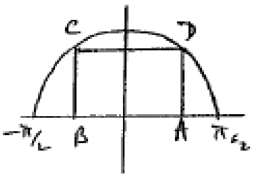
(*c*) $S = \dfrac{2\alpha^2}{\sqrt{(1+\alpha^2)}}$,

(2)

(*d*) $\dfrac{\pi^2}{2\sqrt{(16+\pi^2)}} < S < \sqrt{2}$.

(3)

(a)



By symmetry, B is $(-p, 0)$ M1

Area $= \underline{\underline{2p\cos p}}$ A1 ②

(b) $\frac{dA}{dp} = 2\cos p - 2p\sin p$ B1

$\frac{dA}{dp} = 0 \Rightarrow 1 = p\tan p$, so when $p = \alpha$, $\underline{\alpha\tan\alpha = 1}$ o.e. B1

[This mark can be earned in (c)]

Let $f(p) = p\tan p - 1$ (o.e.)

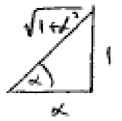
$f(\pi/4) = \pi/4 - 1 < 0$ (f at one end) M1 $(f(\pi/4) < 0)$ A1

$f(1) = \tan 1 - 1 = \tan 1 - \tan \pi/4$

$(1 > \pi/4) \therefore \tan 1 - \tan\pi/4 > 0$ (Reason $f(1) > 0$) A1

change of sign $\therefore$ $\underline{\underline{\pi/4 < \alpha < 1}}$ A1 ⑥

(c) Max. area $= 2\alpha\cos\alpha$, with $\tan\alpha = \frac{1}{\alpha} \Rightarrow$



M1 (Δ or trig)

$= \underline{\underline{\frac{2\alpha^2}{\sqrt{1+\alpha^2}}}}$ $(= S)$ A1 cso ②

(d) $\frac{dS}{d\alpha} = \frac{4\alpha\sqrt{1+\alpha^2} - 2\alpha^3/\sqrt{1+\alpha^2}}{1+\alpha^2} = \frac{4\alpha + 2\alpha^3}{(1+\alpha^2)^{3/2}} > 0$ M1

$\therefore$ S is an increasing function as α varies

S lies between $\frac{2\left(\frac{\pi}{4}\right)^2}{\sqrt{1+\left(\frac{\pi}{4}\right)^2}} = \frac{\pi^2}{2\sqrt{16+\pi^2}}$ (subst. $\alpha = 1$ and $\alpha = \pi/4$) M1

and $\frac{2(1)^2}{\sqrt{1+1}} = \sqrt{2}$

ie $\underline{\frac{\pi^2}{2\sqrt{16+\pi^2}} < S < \sqrt{2}}$ A1 (cso) ③

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2.10 NUMERICAL METHODS (ITERATION, NEWTON-RAPHSON)

No questions available.

2.11 INTEGRATION (EVERYTHING, INCLUDING TRAPEZIUM RULE)

Question 1 (STEP I 2014 Q2)

(i) Show that $\int \ln(2-x)\,\mathrm{d}x = -(2-x)\ln(2-x) + (2-x) + c$, where $x < 2$.

(ii) Sketch the curve A given by $y = \ln|x^2 - 4|$.

(iii) Show that the area of the finite region enclosed by the positive x-axis, the y-axis and the curve A is $4\ln(2+\sqrt{3}) - 2\sqrt{3}$.

(iv) The curve B is given by $y = \big|\ln|x^2-4|\big|$. Find the area between the curve B and the x-axis with $|x| < 2$.

[*Note: you may assume that* $t\ln t \to 0$ *as* $t \to 0$.]

Question 2 (STEP I 2013 Q4)

(i) Show that, for $n > 0$,

$$\int_0^{\frac{1}{4}\pi} \tan^n x \sec^2 x\,\mathrm{d}x = \frac{1}{n+1} \qquad \text{and} \qquad \int_0^{\frac{1}{4}\pi} \sec^n x \tan x\,\mathrm{d}x = \frac{(\sqrt{2})^n - 1}{n}.$$

(ii) Evaluate the following integrals:

$$\int_0^{\frac{1}{4}\pi} x \sec^4 x \tan x\,\mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{1}{4}\pi} x^2 \sec^2 x \tan x\,\mathrm{d}x\,.$$

Solutions: *(ii)* $\frac{\pi}{4} - \frac{1}{3}$ *and* $\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2$

Question 3 (STEP I 2013 Q7)

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Solutions: (ii) $y = x\sqrt{6x^2 - 2x}$ for $x > \frac{1}{3}$

Question 4 (STEP I 2012 Q3)

(i) Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and add to your diagram the tangent to the curve at the origin and the chord joining the origin to the point $(b, \sin b)$, where $0 < b < \frac{1}{2}\pi$.

By considering areas, show that

$$1 - \tfrac{1}{2}b^2 < \cos b < 1 - \tfrac{1}{2}b\sin b\,.$$

(ii) By considering the curve $y = a^x$, where $a > 1$, show that

$$\frac{2(a-1)}{a+1} < \ln a < -1 + \sqrt{2a - 1}\,.$$

[**Hint**: You may wish to write a^x as $\mathrm{e}^{x\ln a}$.]

Question 5 (STEP I 2012 Q5)

Show that

$$\int_0^{\frac{1}{4}\pi} \sin(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{4}(\ln 2 - 1)\,,$$

and that

$$\int_0^{\frac{1}{4}\pi} \cos(2x)\ln(\cos x)\,\mathrm{d}x = \frac{1}{8}(\pi - \ln 4 - 2)\,.$$

Hence evaluate

$$\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} \big(\cos(2x)+\sin(2x)\big)\,\ln\big(\cos x + \sin x\big)\,\mathrm{d}x\,.$$

Question 6 (STEP I 2012 Q8)

(i) Show that substituting $y = xv$, where v is a function of x, in the differential equation

$$xy\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 - 2x^2 = 0 \qquad\qquad (x \neq 0)$$

leads to the differential equation

$$xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 = 0\,.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C\,,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + 6x + 5y = 0 \qquad\qquad (x \neq 0).$$

Question 7 (STEP I 2011 Q2)

The number E is defined by $E = \displaystyle\int_0^1 \frac{\mathrm{e}^x}{1+x}\,\mathrm{d}x$.

Show that

$$\int_0^1 \frac{x\mathrm{e}^x}{1+x}\,\mathrm{d}x = \mathrm{e} - 1 - E\,,$$

and evaluate $\displaystyle\int_0^1 \frac{x^2\mathrm{e}^x}{1+x}\,\mathrm{d}x$ in terms of e and E.

Evaluate also, in terms of E and e as appropriate:

(i) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\frac{1-x}{1+x}}}{1+x}\,\mathrm{d}x\,;$

(ii) $\displaystyle\int_1^{\sqrt{2}} \frac{\mathrm{e}^{x^2}}{x}\,\mathrm{d}x\,.$

Solutions: First part - Use by parts, polynomial division or a suitable substitution.
Second part: $2 - e + E$
(i) E (ii) $\frac{eE}{2}$

Question 8 (STEP I 2011 Q5)

Given that $0 < k < 1$, show with the help of a sketch that the equation

$$\sin x = kx \qquad (*)$$

has a unique solution in the range $0 < x < \pi$.

Let

$$I = \int_0^{\pi} \left| \sin x - kx \right| \mathrm{d}x\,.$$

Show that

$$I = \frac{\pi^2 \sin\alpha}{2\alpha} - 2\cos\alpha - \alpha\sin\alpha\,,$$

where α is the unique solution of $(*)$.

Show that I, regarded as a function of α, has a unique stationary value and that this stationary value is a minimum. Deduce that the smallest value of I is

$$-2\cos\frac{\pi}{\sqrt{2}}\,.$$

Question 9 (STEP 2010 Q4)

Use the substitution $x = \dfrac{1}{t^2-1}$, where $t > 1$, to show that, for $x > 0$,

$$\int \frac{1}{\sqrt{x\,(x+1)}}\,\mathrm{d}x = 2\ln\left(\sqrt{x} + \sqrt{x+1}\,\right) + c\,.$$

[**Note:** You may use without proof the result $\displaystyle\int \frac{1}{t^2-a^2}\,\mathrm{d}t = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + \text{constant.}$]

The section of the curve

$$y = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}$$

between $x = \frac{1}{8}$ and $x = \frac{9}{16}$ is rotated through 360^o about the x-axis. Show that the volume enclosed is $2\pi\ln\frac{5}{4}$.

Question 10 (STEP 2010 Q6)

Show that, if $y = \mathrm{e}^x$, then

$$(x-1)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0\,. \qquad (*)$$

In order to find other solutions of this differential equation, now let $y = u\mathrm{e}^x$, where u is a function of x. By substituting this into $(*)$, show that

$$(x-1)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + (x-2)\frac{\mathrm{d}u}{\mathrm{d}x} = 0\,. \qquad (**)$$

By setting $\dfrac{\mathrm{d}u}{\mathrm{d}x} = v$ in $(**)$ and solving the resulting first order differential equation for v, find u in terms of x. Hence show that $y = Ax + B\mathrm{e}^x$ satisfies $(*)$, where A and B are any constants.

Question 11 (STEP 2009 Q6)

(i) Show that, for $m > 0$,

$$\int_{1/m}^{m} \frac{x^2}{x+1}\,\mathrm{d}x = \frac{(m-1)^3(m+1)}{2m^2} + \ln m\,.$$

(ii) Show by means of a substitution that

$$\int_{1/m}^{m} \frac{1}{x^n(x+1)}\,\mathrm{d}x = \int_{1/m}^{m} \frac{u^{n-1}}{u+1}\,\mathrm{d}u\,.$$

(iii) Evaluate:

(a) $\displaystyle\int_{1/2}^{2} \frac{x^5+3}{x^3(x+1)}\,\mathrm{d}x$;

(b) $\displaystyle\int_{1}^{2} \frac{x^5+x^3+1}{x^3(x+1)}\,\mathrm{d}x$.

Solution: (iii)(a) $\frac{3}{2} + 4ln\ 2$ (b) $\frac{3}{8} + ln\ 3$

Question 12 (STEP 2009 Q7)

Show that, for any integer m,

$$\int_{0}^{2\pi} \mathrm{e}^x \cos mx\,\mathrm{d}x = \frac{1}{m^2+1}(\mathrm{e}^{2\pi}-1)\,.$$

(i) Expand $\cos(A+B) + \cos(A-B)$. Hence show that

$$\int_{0}^{2\pi} \mathrm{e}^x \cos x \cos 6x\,\mathrm{d}x = \tfrac{19}{650}(\mathrm{e}^{2\pi}-1)\,.$$

(ii) Evaluate $\displaystyle\int_{0}^{2\pi} \mathrm{e}^x \sin 2x \sin 4x \cos x\,\mathrm{d}x$.

Solution: (ii) $\frac{44}{325}(e^{2\pi} - 1)$

Question 13 (STEP 2008 Q6)

The function f is defined by

$$\mathrm{f}(x) = \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1}, \qquad x \geqslant 0,$$

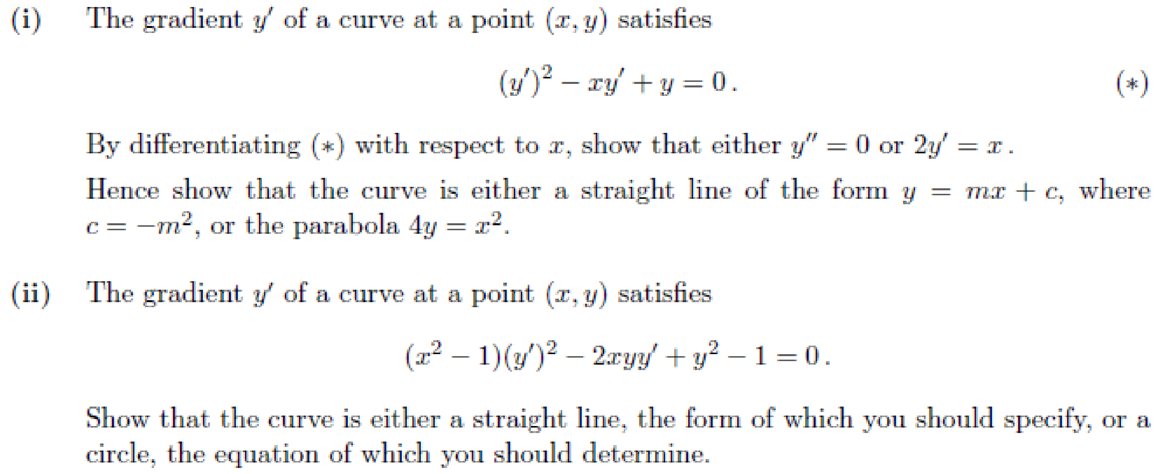
and the function g is the inverse function to f, so that $\mathrm{g}(\mathrm{f}(x)) = x$. Sketch $\mathrm{f}(x)$ and $\mathrm{g}(x)$ on the same axes.

Verify, by evaluating each integral, that

$$\int_{0}^{\frac{1}{2}} \mathrm{f}(x)\,\mathrm{d}x + \int_{0}^{k} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{2(\sqrt{\mathrm{e}}+1)}\,,$$

where $k = \dfrac{1}{\sqrt{\mathrm{e}}+1}$, and explain this result by means of a diagram.

Question 14 (STEP 2008 Q8)



(i) The gradient y' of a curve at a point (x, y) satisfies

$$(y')^2 - xy' + y = 0 . \qquad (*)$$

By differentiating $(*)$ with respect to x, show that either $y'' = 0$ or $2y' = x$.

Hence show that the curve is either a straight line of the form $y = mx + c$, where $c = -m^2$, or the parabola $4y = x^2$.

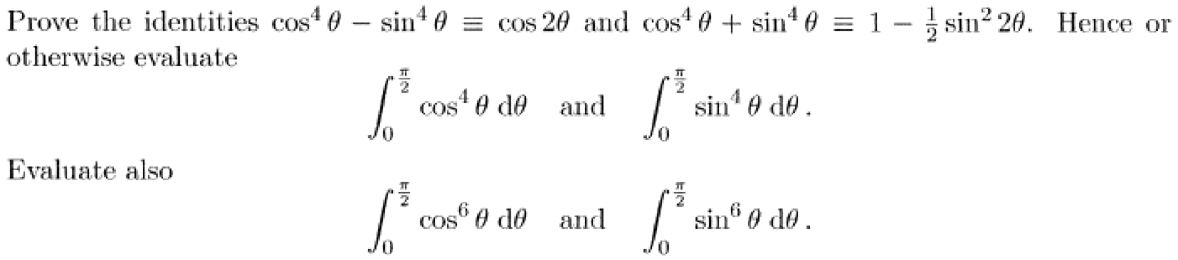
(ii) The gradient y' of a curve at a point (x, y) satisfies

$$(x^2 - 1)(y')^2 - 2xyy' + y^2 - 1 = 0 .$$

Show that the curve is either a straight line, the form of which you should specify, or a circle, the equation of which you should determine.

Solution: (ii) $x^2 + y^2 = 1$

Question 15 (STEP 2007 Q3)



Prove the identities $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$ and $\cos^4\theta + \sin^4\theta \equiv 1 - \frac{1}{2}\sin^2 2\theta$. Hence or otherwise evaluate

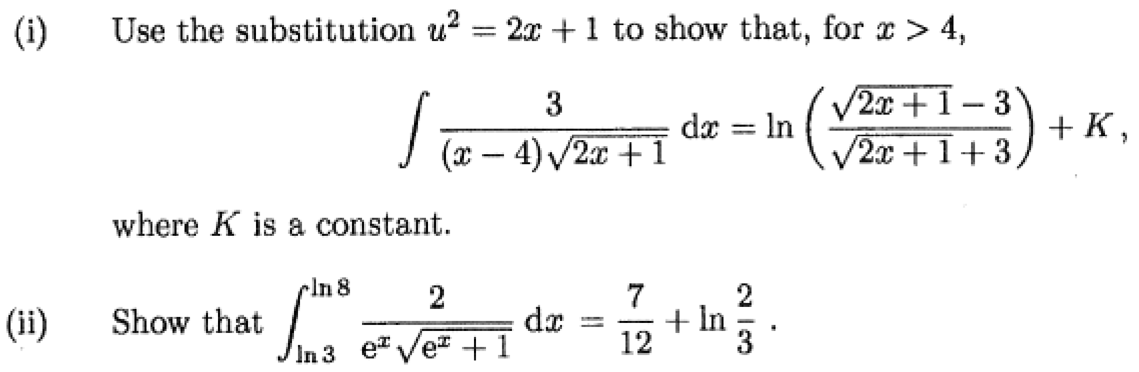
$$\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta .$$

Evaluate also

$$\int_0^{\frac{\pi}{2}} \cos^6\theta \, d\theta \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^6\theta \, d\theta .$$

Solutions: (i) Both $\frac{3\pi}{16}$ (ii) Both $\frac{5\pi}{32}$

Question 16 (STEP 2006 Q5)



(i) Use the substitution $u^2 = 2x + 1$ to show that, for $x > 4$,

$$\int \frac{3}{(x-4)\sqrt{2x+1}} \, dx = \ln\left(\frac{\sqrt{2x+1} - 3}{\sqrt{2x+1} + 3}\right) + K ,$$

where K is a constant.

(ii) Show that $\displaystyle\int_{\ln 3}^{\ln 8} \frac{2}{e^x\sqrt{e^x + 1}} \, dx = \frac{7}{12} + \ln\frac{2}{3}$.

Question 17 (STEP 2006 Q7)

(i) Sketch on the same axes the functions $\operatorname{cosec} x$ and $2x/\pi$, for $0 < x < \pi$. Deduce that the equation $x \sin x = \pi/2$ has exactly two roots in the interval $0 < x < \pi$.

Show that

$$\int_{\pi/2}^{\pi} \left| x \sin x - \frac{\pi}{2} \right| \, \mathrm{d}x = 2\sin\alpha + \frac{3\pi^2}{4} - \alpha\pi - \pi - 2\alpha\cos\alpha - 1$$

where α is the larger of the roots referred to above.

(ii) Show that the region bounded by the positive x-axis, the y-axis and the curve $y = \Big| |\mathrm{e}^x - 1| - 1 \Big|$ has area $\ln 4 - 1$.

Question 18 (STEP 2005 Q5)

(i) Evaluate the integral

$$\int_0^1 (x+1)^{k-1} \, \mathrm{d}x$$

in the cases $k \neq 0$ and $k = 0$.

Deduce that $\dfrac{2^k - 1}{k} \approx \ln 2$ when $k \approx 0$.

(ii) Evaluate the integral

$$\int_0^1 x\,(x+1)^m \, \mathrm{d}x$$

in the different cases that arise according to the value of m.

Solutions: (i) When $k \neq 0, I = \frac{2^k - 1}{k}$ When $k = 0, I = ln\ 2$

(ii) If $m \neq -1, -2,\ \ I = \frac{m2^{m+1}+1}{(m+2)(m+1)}$ If $m = -1, I = 1 - \frac{1}{x+1}$

If $m = -2, I = ln\ 2 - \frac{1}{2}$

Question 19 (STEP I 2005 Q8)

Show that, if $y^2 = x^k \mathrm{f}(x)$, then $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = ky^2 + x^{k+1}\dfrac{\mathrm{df}}{\mathrm{d}x}$.

(i) By setting $k = 1$ in this result, find the solution of the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + x^2 - 1$$

for which $y = 2$ when $x = 1$. Describe geometrically this solution.

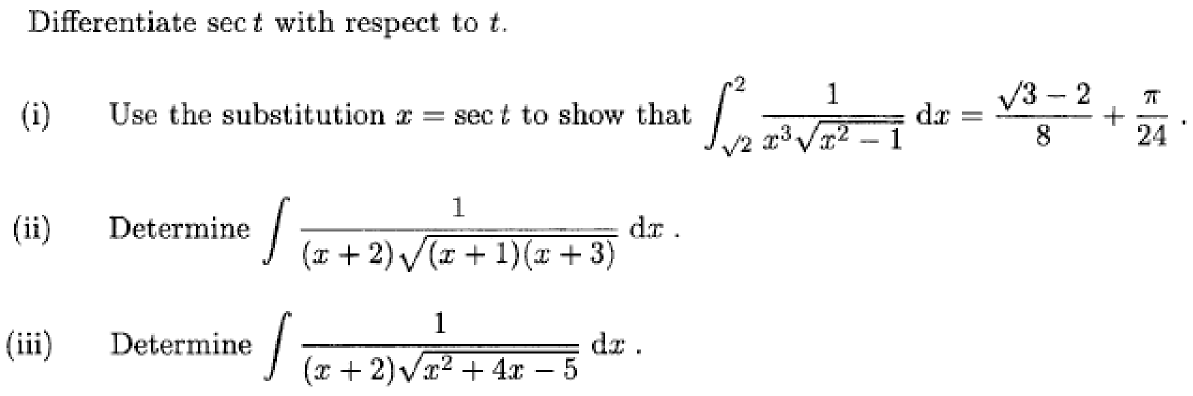
(ii) Find the solution of the differential equation

$$2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2\ln(x) - xy^2$$

for which $y = 1$ when $x = 1$.

Solutions: (i) $y = \pm(x+1)$ (ii) $y^2 = \frac{(\ln x)^2 + 1}{x}$

Question 20 (STEP I 2004 Q4)



Differentiate $\sec t$ with respect to t.

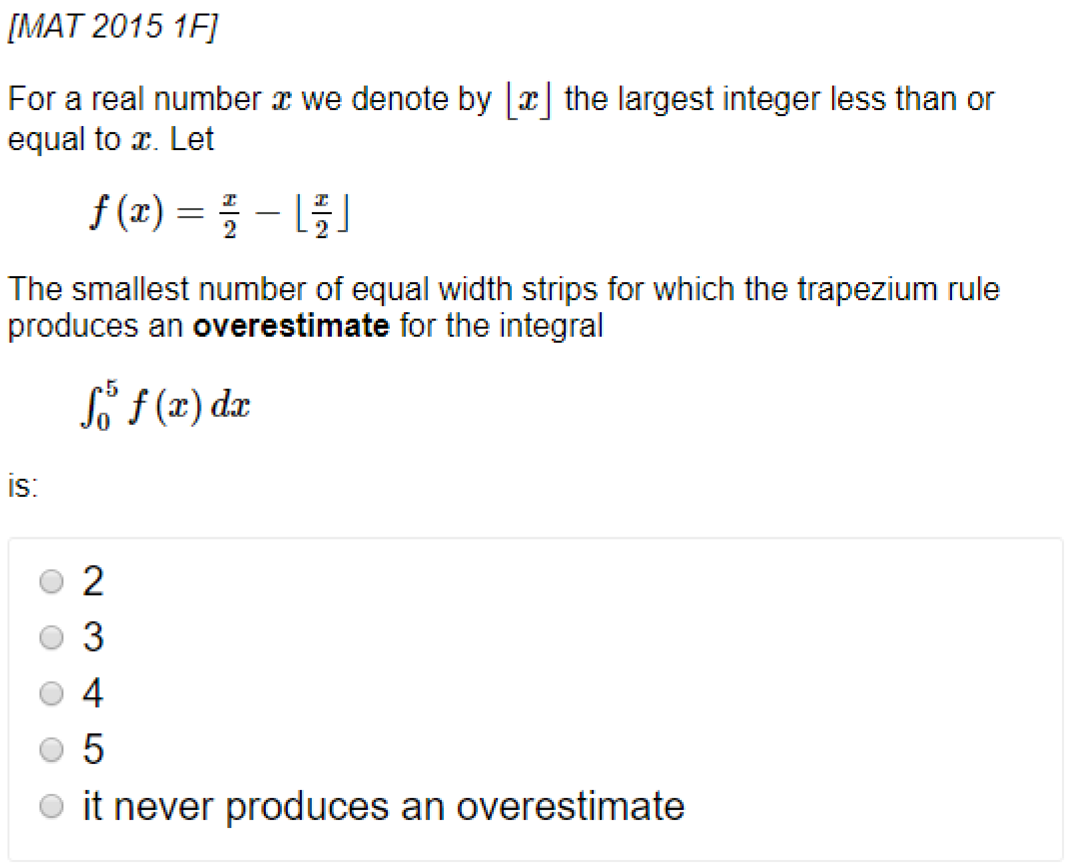
(i) Use the substitution $x = \sec t$ to show that $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,\mathrm{d}x = \frac{\sqrt{3}-2}{8} + \frac{\pi}{24}$.

(ii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{(x+1)(x+3)}}\,\mathrm{d}x$.

(iii) Determine $\displaystyle\int \frac{1}{(x+2)\sqrt{x^2+4x-5}}\,\mathrm{d}x$.

Solutions: (ii) $arsec(x+2)+c$ (iii) $\frac{1}{3}\text{arcsec}\left(\frac{x+2}{3}\right)+c$

Question 21



[MAT 2015 1F]

For a real number x we denote by $\lfloor x \rfloor$ the largest integer less than or equal to x. Let

$$f(x) = \tfrac{x}{2} - \lfloor \tfrac{x}{2} \rfloor$$

The smallest number of equal width strips for which the trapezium rule produces an **overestimate** for the integral

$$\int_0^5 f(x)\,dx$$

is:

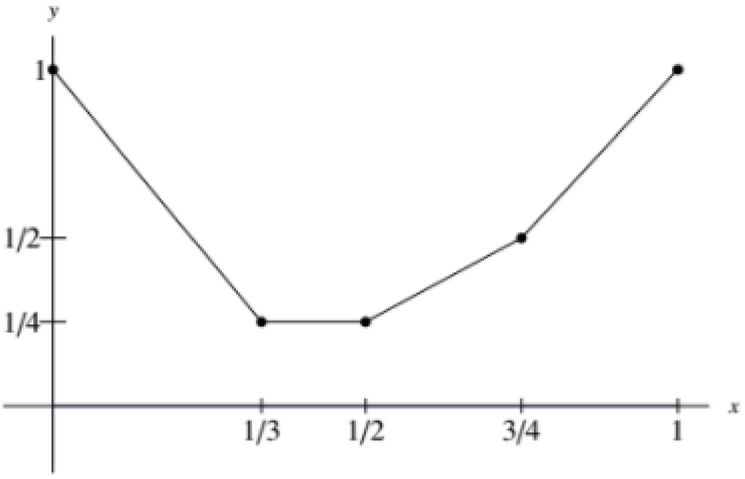
○ 2
○ 3
○ 4
○ 5
○ it never produces an overestimate

Solution: 3

Question 22

[MAT 2010 1F]

The graph $y = f(x)$ of a function is drawn below for $0 \leq x \leq 1$



The trapezium rule is then used to estimate

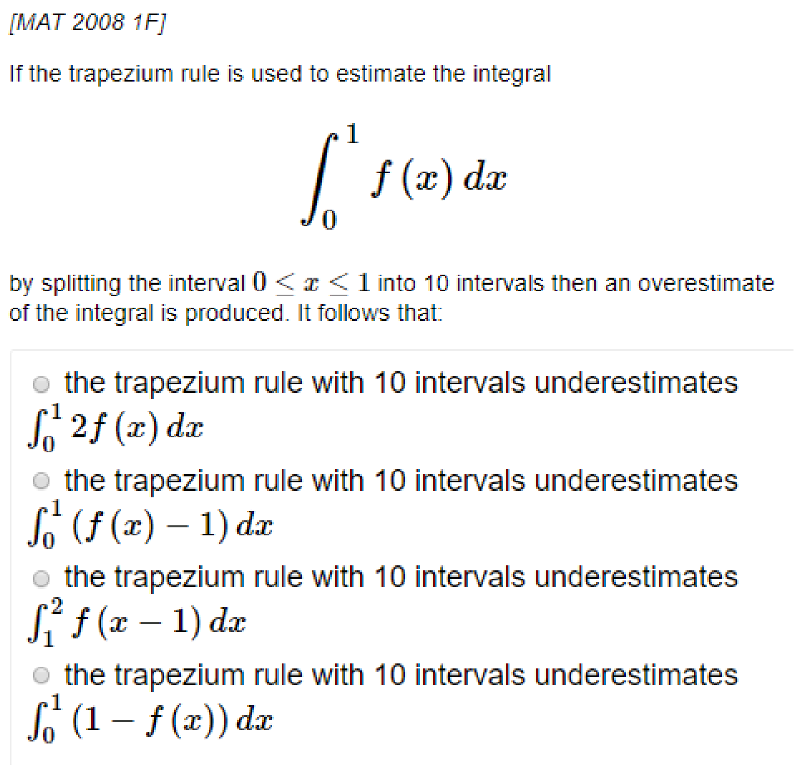
$$\int_0^1 f(x)\,dx$$

by dividing $0 \leq x \leq 1$ into n equal intervals. The estimate calculated will equal the actual integral when

- ○ n is a multiple of 4;
- ○ n is a multiple of 6;
- ○ n is a multiple of 8;
- ○ n is a multiple of 12;

Solution: n is a multiple of 12

Question 23



[MAT 2008 1F]

If the trapezium rule is used to estimate the integral

$$\int_0^1 f(x)\,dx$$

by splitting the interval $0 \leq x \leq 1$ into 10 intervals then an overestimate of the integral is produced. It follows that:

○ the trapezium rule with 10 intervals underestimates $\int_0^1 2f(x)\,dx$

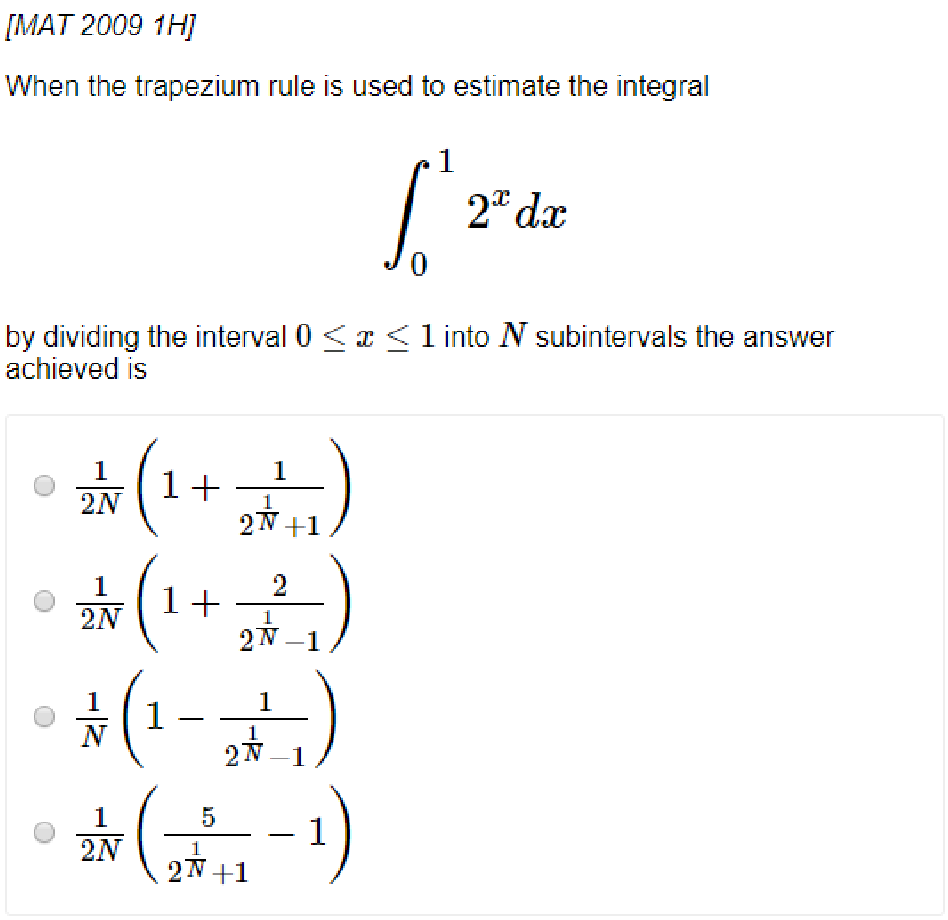
○ the trapezium rule with 10 intervals underestimates $\int_0^1 (f(x)-1)\,dx$

○ the trapezium rule with 10 intervals underestimates $\int_1^2 f(x-1)\,dx$

○ the trapezium rule with 10 intervals underestimates $\int_0^1 (1-f(x))\,dx$

Solution: Option 4

Question 24



[MAT 2009 1H]

When the trapezium rule is used to estimate the integral

$$\int_0^1 2^x\,dx$$

by dividing the interval $0 \leq x \leq 1$ into N subintervals the answer achieved is

○ $\frac{1}{2N}\left(1+\frac{1}{2^{\frac{1}{N}}+1}\right)$

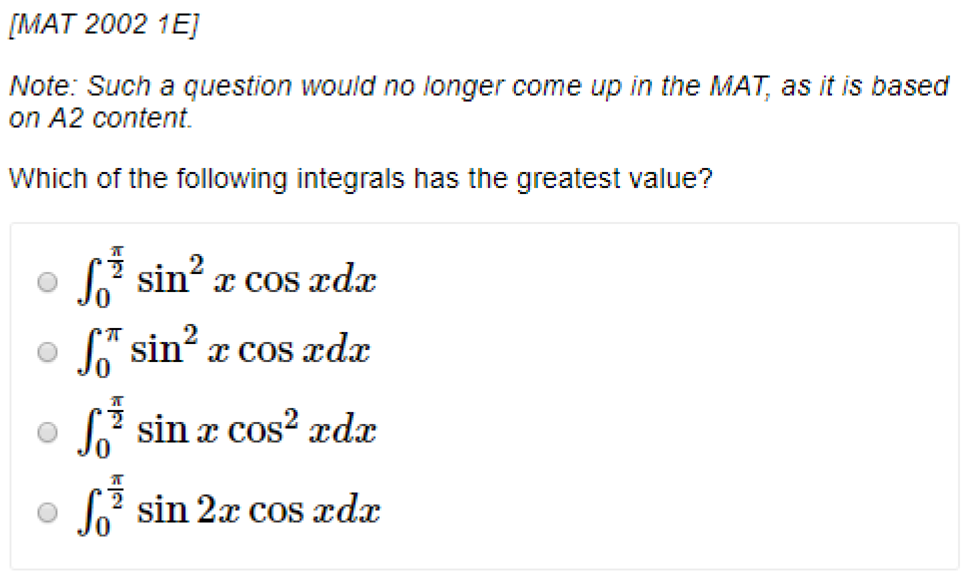
○ $\frac{1}{2N}\left(1+\frac{2}{2^{\frac{1}{N}}-1}\right)$

○ $\frac{1}{N}\left(1-\frac{1}{2^{\frac{1}{N}}-1}\right)$

○ $\frac{1}{2N}\left(\frac{5}{2^{\frac{1}{N}}+1}-1\right)$

Solution: Option 2

Question 25



[MAT 2002 1E]

Note: Such a question would no longer come up in the MAT, as it is based on A2 content.

Which of the following integrals has the greatest value?

○ $\int_0^{\frac{\pi}{2}} \sin^2 x \cos x dx$

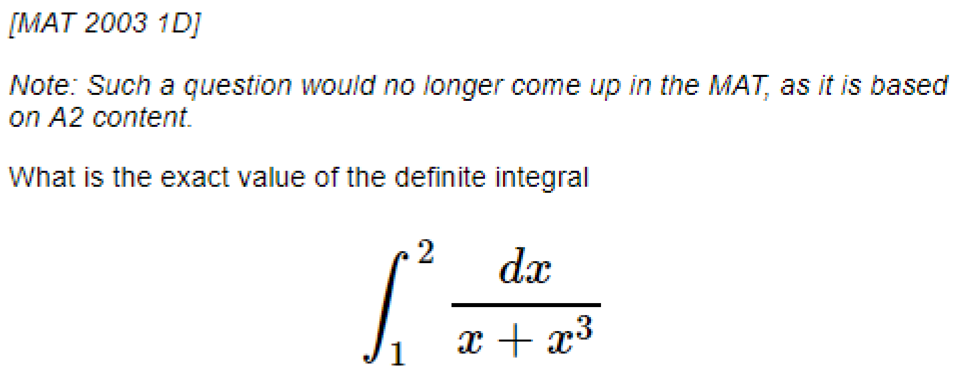
○ $\int_0^{\pi} \sin^2 x \cos x dx$

○ $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x dx$

○ $\int_0^{\frac{\pi}{2}} \sin 2x \cos x dx$

Solution: Option 4

Question 26



[MAT 2003 1D]

Note: Such a question would no longer come up in the MAT, as it is based on A2 content.

What is the exact value of the definite integral

$$\int_1^2 \frac{dx}{x + x^3}$$

Solution: $\frac{1}{2} \ln \left(\frac{8}{5}\right)$

Question 27 (AEA 2013 Q5)

In this question u and v are functions of x. Given that $\int \mathrm{u}\,\mathrm{d}x$, $\int \mathrm{v}\,\mathrm{d}x$ and $\int \mathrm{uv}\,\mathrm{d}x$ satisfy

$$\int \mathrm{uv}\,\mathrm{d}x = \left(\int \mathrm{u}\,\mathrm{d}x\right)\times\left(\int \mathrm{v}\,\mathrm{d}x\right) \qquad \mathrm{uv} \neq 0$$

(a) show that $1 = \dfrac{\int \mathrm{u}\,\mathrm{d}x}{\mathrm{u}} + \dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}}$ **(3)**

Given also that $\dfrac{\int \mathrm{u}\,\mathrm{d}x}{\mathrm{u}} = \sin^2 x$,

(b) use part (a) to write down an expression, in terms of x, for $\dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}}$, **(1)**

(c) show that

$$\frac{1}{\mathrm{u}}\frac{\mathrm{du}}{\mathrm{d}x} = \frac{1-2\sin x\cos x}{\sin^2 x}$$ **(3)**

(d) hence use integration to show that $\mathrm{u} = Ae^{-\cot x}\mathrm{cosec}^2 x$, where A is an arbitrary constant. **(6)**

(e) By differentiating $e^{\tan x}$ find a similar expression for v. **(2)**

Question	Scheme	Marks	Notes
(a)	Differentiate: $\mathrm{uv} = \mathrm{v}\int \mathrm{u}\,\mathrm{d}x + \mathrm{u}\int \mathrm{v}\,\mathrm{d}x$	M1 A1	Attempt to diff Correct prod. rule
	$\div\,\mathrm{uv}$ leading to $1 = \dfrac{\int \mathrm{u}\,\mathrm{d}x}{\mathrm{u}} + \dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}}$ (*)	A1cso (3)	
(b)	$\dfrac{\int \mathrm{v}\,\mathrm{d}x}{\mathrm{v}} = \cos^2 x$	B1 (1)	S+ for $1 - c^2 = s^2$
(c)	Diff. $\mathrm{u}\sin^2 x = \int \mathrm{u}\,\mathrm{d}x$ gives $\mathrm{u} = \dfrac{\mathrm{du}}{\mathrm{d}x}\sin^2 x + \mathrm{u}2\sin x\cos x$	M1	Multiply by u and differentiate Or quotient rule Collect u terms
	$\dfrac{\mathrm{du}}{\mathrm{d}x}\sin^2 x = \mathrm{u}(1-2\sin x\cos x)$ $\therefore \dfrac{1}{\mathrm{u}}\dfrac{\mathrm{du}}{\mathrm{d}x} = \dfrac{1-2\sin x\cos x}{\sin^2 x}$	dM1 A1cso (3)	
(d)	Separate variables: $\int \dfrac{1}{\mathrm{u}}\mathrm{du} = \int\left(\dfrac{1-2\sin x\cos x}{\sin^2 x}\right)\,\mathrm{d}x$	M1	Separation of vars. Condone missing integral signs.
	RHS $= \int\left(\mathrm{cosec}^2 x - 2\cot x\right)\,\mathrm{d}x$	M1	Prepares RHS
	Integrate: $\ln \mathrm{u} = -\cot x, -2\ln\sin x \quad +c$	A1,A1	$+c$ on 2nd A1
	$\ln\left(\mathrm{u}\sin^2 x\right) = -\cot x \quad (+c)$	M1	Collect ln terms or remove ln
	$\mathrm{u} = Ae^{-\cot x}\mathrm{cosec}^2 x$	A1cso (6)	No incorrect work
(c)	$y = e^{\tan x} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{\tan x}\sec^2 x$ or $e^{\tan x}\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan x)$	M1	For differentiation
	Hence $\mathrm{v} = Be^{\tan x}\sec^2 x$	A1 (2) **(15)**	Condone A not B but S-

Question 28 (AEA 2013 Q6)

(a) Starting from $\left[\mathrm{f}(x)-\lambda \mathrm{g}(x)\right]^2 \geqslant 0$ show that λ satisfies the quadratic inequality

$$\left(\int_a^b [\mathrm{g}(x)]^2 \,\mathrm{d}x\right)\lambda^2 - 2\left(\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right)\lambda + \int_a^b [\mathrm{f}(x)]^2 \,\mathrm{d}x \geqslant 0$$

where a and b are constants and λ can take any real value.

(2)

(b) Hence prove that

$$\left[\int_a^b \mathrm{f}(x)\mathrm{g}(x)\,\mathrm{d}x\right]^2 \leqslant \left[\int_a^b [\mathrm{f}(x)]^2 \,\mathrm{d}x\right] \times \left[\int_a^b [\mathrm{g}(x)]^2 \,\mathrm{d}x\right]$$

(3)

(c) By letting $\mathrm{f}(x) = 1$ and $\mathrm{g}(x) = \left(1+x^3\right)^{\frac{1}{2}}$ show that

$$\int_{-1}^{2} \left(1+x^3\right)^{\frac{1}{2}} \,\mathrm{d}x \leqslant \frac{9}{2}$$

(4)

(d) Show that $\int_{-1}^{2} x^2\left(1+x^3\right)^{\frac{1}{4}} \,\mathrm{d}x = \frac{12\sqrt{3}}{5}$

(3)

(e) Hence show that

$$\frac{144}{55} \leqslant \int_{-1}^{2} \left(1+x^3\right)^{\frac{1}{2}} \,\mathrm{d}x$$

(4)

Question	Scheme	Marks	Notes
(a)	$\left[\mathrm{f}(x)-\lambda \mathrm{g}(x)\right]^2=\left[\mathrm{f}(x)\right]^2-2\lambda \mathrm{f}(x)\mathrm{g}(x)+\lambda^2\left[\mathrm{g}(x)\right]^2$	M1	Attempt to multiply
S+ for area comment	Integrate dx throughout with inequality	A1cso	No incorrect work
		(2)	
(b)	Treat as quadratic in λ and attempt to use discriminant	M1	Δ & identify *a, b, c*
	Clear reason for use of $b^2-4ac\le 0$ (or <0) e.g. "no real roots"	M1	Reason for ≤ 0
	Giving: $\left[\int \mathrm{f}(x)\mathrm{g}(x)\ \mathrm{d}x\right]^2\le\left[\int[\mathrm{f}(x)]^2\ \mathrm{d}x\right]\times\left[\int[\mathrm{g}(x)]^2\ \mathrm{d}x\right]$ (o.e.)	A1cso	Condone 4s
		(3)	
(c)	$\mathrm{g}(x)=\left(1+x^3\right)^{\frac{1}{2}}$ and $\mathrm{f}(x)=1$		
	Then $[E]^2\le\left[\int\left(1+x^3\right)\ \mathrm{d}x\right]\times\left[\int 1^2\ \mathrm{d}x\right]$	M1	
	$\int_{-1}^{2}\left(1+x^3\right)\ \mathrm{d}x=\left[x+\frac{x^4}{4}\right]_{-1}^{2}=,(2+4)-\left(-1+\tfrac{1}{4}\right)=\frac{27}{4}$	M1, A1	Integration 6.75 (o.e.)
	So $E^2\le\frac{81}{4}$ i.e. $E\le\frac{9}{2}$	A1cso	
		(4)	
(d)	$\int x^2\left(1+x^3\right)^{\frac{1}{4}}\ \mathrm{d}x=\frac{4}{15}\left(1+x^3\right)^{\frac{5}{4}}$	M1 A1	k(..) and 5/4 power All correct
	$\left\{\left[\frac{4}{15}\left(1+x^3\right)^{\frac{5}{4}}\right]_{-1}^{2}=\right\}\frac{4}{15}\left[(9)^{\frac{5}{4}}-0\right]=\frac{4}{15}\times 9\sqrt{3}=\frac{12\sqrt{3}}{5}$	A1cso	Must see one of the expr' between {..} and the answer
(e)	Let E = required integral.		
	$\mathrm{f}(x)=\left(1+x^3\right)^{\frac{1}{4}}$ and $\mathrm{g}(x)=x^2$	B1	Suitable f and g
	Then $[(\mathrm{d})]^2\le E\times\int_{-1}^{2}x^4\ \mathrm{d}x$	M1	Suitable inequality for *E*
	$\int_{-1}^{2}x^4\ \mathrm{d}x=\left[\frac{x^5}{5}\right]_{-1}^{2}=\frac{32}{5}--\frac{1}{5}=\frac{33}{5}$	M1	Allow slip e.g $\frac{16}{5}--\frac{1}{5}$ or $\frac{32}{5}-\frac{1}{5}$
	So $\frac{144\times 3}{25}\le E\times\frac{33}{5}\rightarrow E\ge\frac{144}{55}$	A1cso	
		(4) **(16)**	

Question 29 (AEA 2012 Q2)

(a) Show that

$$\sin 3x = 3\sin x - 4\sin^3 x$$

(3)

Hence find

(b) $\int \cos x(6\sin x - 2\sin 3x)^{\frac{2}{3}}\ \mathrm{d}x$

(3)

(c) $\int (3\sin 2x - 2\sin 3x\cos x)^{\frac{1}{3}}\ \mathrm{d}x$

(4)

Qu	Scheme	Mark	Notes
(a)	$\sin(2x+x) = \sin 2x\cos x + \cos 2x\sin x$	M1	Use of sin $(A+B)$
	$= 2\sin x\cos^2 x + (\sin x - 2\sin^3 x)$	M1	Use of sin2x and cos2x
	$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x = 3\sin x - 4\sin^3 x$	A1cso	
Use of i	$\sin 3x = 3\cos^2 x\sin x, -\sin^3 x$ for M1, M1	**(3)**	
(b)	$6\sin x - 2\sin 3x = 6\sin x - 2(3\sin x - 4\sin^3 x) = [8\sin^3 x]$	M1	Attempt to use (a)
	$I = \int \cos x 4\sin^2 x\ \mathrm{d}x$	A1	For 4sin^2xcos$x$ only
	$= \underline{\frac{4\sin^3 x}{3}}$ $(+c)$ (o.e.) e.g. $\frac{2}{3}\sin 2x\cos x - \frac{4}{3}\sin x\cos 2x$ $(+c)$	A1 **(3)**	
(c)	$\int(3\sin 2x - 2\sin 3x\cos x)^{\frac{1}{3}}\mathrm{d}x = \int(6\sin x\cos x - 2\sin 3x\cos x)^{\frac{1}{3}}\ \mathrm{d}x$	M1	Use of sin2x
	$= \int \cos^{\frac{1}{3}} x 2\sin x\ \mathrm{d}x$ <u>or</u> $\int(8\cos x\sin^3 x)^{\frac{1}{3}}\ \mathrm{d}x$	A1	Use of (a) to simplify integrand
	$= \underline{-\frac{3}{2}\cos^{\frac{4}{3}} x}$ $(+c)$	M1 A1 **(4)**	Attempt int. $\rightarrow k\cos^{\frac{4}{3}} x$

Question 30 (AEA 2012 Q6)

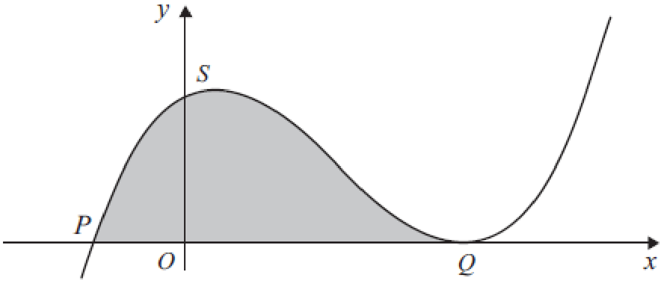


Figure 1

Figure 1 shows a sketch of the curve with equation $y = (x + a)(x - b)^2$, where a and b are positive constants. The curve cuts the x-axis at P and has a maximum point at S and a minimum point at Q.

(a) Write down the coordinates of P and Q in terms of a and b.

(2)

(b) Show that G, the area of the shaded region between the curve PSQ and the x-axis, is given by $G = \dfrac{(a+b)^4}{12}$.

(6)

The rectangle $PQRST$ has RST parallel to QP and both PT and QR are parallel to the y-axis.

(c) Show that $\dfrac{G}{\text{Area of } PQRST} = k$, where k is a constant independent of a and b and find the value of k.

(8)

. **(a)**	$P\ (-a,\ 0)\quad Q\ (b{,}0)$	B1B1	Allow B1B0 for (0, -a) etc
		(2)	
(b)	$I = \int (x+a)\ \mathrm{d}\left[\dfrac{(x-b)^3}{3}\right], = \left[(x+a)\dfrac{(x-b)^3}{3}\right]_{-a}^{b} - \int \dfrac{(x-b)^3}{3}\ \mathrm{d}x$	M1, A1-A1	M1 for correct attempt by parts
	$= 0, -\left[\dfrac{(x-b)^4}{12}\right]_{-a}^{b} = (0) - -\dfrac{(-a-b)^4}{12} = \dfrac{(a+b)^4}{12}$	B1, M1 A1cso	M1 for second stage integration
		(6)	
(c)	$y' = (x-b)^2 + (x+a)2(x-b)$	M1	Some correct diff'n
	$y' = 0 \Rightarrow 0 = (x-b)[x-b+2x+2a]$	M1	Attempt to solve $y' = 0$
	$x = \dfrac{b-2a}{3}$	A1	
	y co-ord of S is: $y_S = \dfrac{(a+b)}{3}\left(\dfrac{-2a-2b}{3}\right)^2 = \dfrac{4}{27}(a+b)^3$	dM1	Sub to get y co-ord of S Dep on 2nd M1
	Area of $PQRST = y_S \times (a+b), = \dfrac{4}{27}(a+b)^4$	dM1A1	M1 using correct formula Dep on 3rd M1
	Ratio $= \dfrac{\dfrac{(a+b)^4}{12}}{\dfrac{4}{27}(a+b)^4},\ = \underline{\dfrac{27}{48} = \dfrac{9}{16}}$	dM1,A1 **(8)**	M1 dep on 2nd and 3rd M1. Must eliminate $(a+b)^4$
		[16]	
ALT **(b)**	<u>Expand</u> $I = \int \left(x^3 + ax^2 - 2bx^2 - 2abx + b^2x + ab^2\right)\ \mathrm{d}x$	M1A1	M1 for 6 terms (3 corr) A1 for all correct
	$= \left(\dfrac{b^4}{12} + \dfrac{4ab^3}{12}\right) - \left(-\dfrac{a^4}{12} - \dfrac{4a^3b}{12} - \dfrac{6a^2b^2}{12}\right) \rightarrow$ answer	M1B1 A1 A1cso	M1 some integration B1 some use of b & $-a$ A1 one bracket correct

Question 31 (AEA 2012 Q7)

[arccos$x$ and arctan$x$ are alternative notation for $\cos^{-1}x$ and $\tan^{-1}x$ respectively]

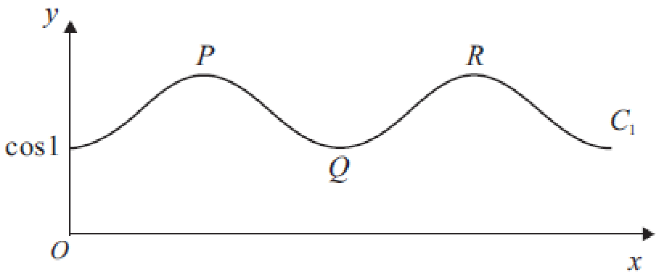


Figure 2

Figure 2 shows a sketch of the curve C_1 with equation $y = \cos(\cos x)$, $0 \leqslant x < 2\pi$.

The curve has turning points at (0, cos1), P, Q and R as shown in Figure 2.

(a) Find the coordinates of the points P, Q and R.

(4)

The curve C_2 has equation $y = \sin(\cos x)$, $0 \leqslant x < 2\pi$. The curves C_1 and C_2 intersect at the points S and T.

(b) Copy Figure 2 and on this diagram sketch C_2 stating the coordinates of the minimum point on C_2 and the points where C_2 meets or crosses the coordinate axes.

(5)

The coordinates of S are (α, d) where $0 < \alpha < \pi$.

(c) Show that $\alpha = \arccos\left(\frac{\pi}{4}\right)$.

(2)

(d) Find the value of d in surd form and write down the coordinates of T.

(3)

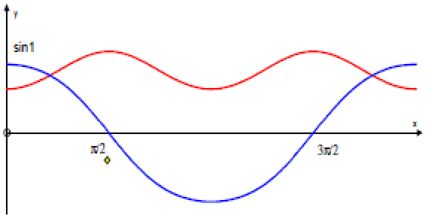
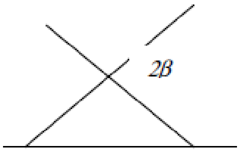
The tangent to C_1 at the point S has gradient $\tan\beta$.

(e) Show that $\beta = \arctan\sqrt{\left(\frac{16-\pi^2}{32}\right)}$.

(5)

(f) Find, in terms of β, the obtuse angle between the tangent to C_1 at S and the tangent to C_2 at S.

(5)

(a)	Max of $\cos u$ is 1 when $u = 0$, $u = \cos x = 0$ when $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$	M1	Method to get at least one of these values Implied by correct P or R
	$P(\frac{\pi}{2},1)$ $R(\frac{3\pi}{2},1)$ [Require 1 not cos(0)]	A1A1	**Condone degrees in any part**
	$\cos(-1) = \cos(1)$ so $Q(\pi, \cos 1)$ [Accept cos(-1)]	B1 **(4)**	
(b)		B1	Shape (one –ve min)
		B1	sin1 seen at ends and cos1< sin1 < 1
	Accept points NOT marked on graph	B1,B1	$\frac{\pi}{2}, \frac{3\pi}{2}$
		B1 **(5)**	$(\pi, \sin(-1))$
(c)	$\cos(\cos x) = \sin(\cos x) \Rightarrow 1 = \tan(\cos x)$	M1	Use of sin/cos= tan
	$\cos x = \frac{\pi}{4}$ (or $\frac{5\pi}{4}$) so $x = \alpha = \arccos\left(\frac{\pi}{4}\right)$	A1cso **(2)**	Allow verify but needs a comment " so $\alpha = \ldots$"
(d)	$d = \cos(\cos\alpha) = \cos\left(\frac{\pi}{4}\right)$	M1	
	$S\left(\arccos\left(\frac{\pi}{4}\right), \frac{1}{\sqrt{2}}\right)$ Accept $d = \frac{1}{\sqrt{2}}$ (o.e.)	A1	
	$T\left(2\pi - \arccos\left(\frac{\pi}{4}\right), \frac{1}{\sqrt{2}}\right)$	B1ft **(3)**	ft their y co-ord of S
(e)	$y' = \sin(\cos x)\sin x$	M1A1	M1 for attempt at chain rule
	$m = \sin\left(\frac{\pi}{4}\right)\sin\alpha$	M1	Substitution attempt
	$m = \frac{1}{\sqrt{2}} \times \frac{\sqrt{16-\pi^2}}{4}$	M1	Attempt $\sin\alpha$ in π
	$m = \sqrt{\frac{16-\pi^2}{32}}$ so $\beta = \arctan\left(\sqrt{\frac{16-\pi^2}{32}}\right)$	A1cso **(5)**	
(f)	For C_2 : $y' = -\cos(\cos x)\sin x$	M1	Attempt y'
	$m' = -\cos\left(\frac{\pi}{4}\right)\sin\alpha, \quad = -\tan\beta$ (o.e.) e.g. $-\sqrt{\frac{16-\pi^2}{32}}$	M1A1	M1 for sub of α
		M1	Attempt to find angle between two tangents to get 2β or $\pi - 2\beta$
	Obtuse angle is $\pi - 2\beta$	A1 (5)	Allow $180 - 2\beta$
	[$\tan\beta = \sqrt{\frac{16-\pi^2}{32}} < 1 \Rightarrow \beta < \frac{\pi}{4}$ so 2β is acute for S+]	**[24]**	

Question 32 (AEA 2011 Q2)

Given that

$$\int_0^{\frac{\pi}{2}} \left(1+\tan\left[\tfrac{1}{2}x\right]\right)^2 \, dx = a + \ln b$$

find the value of a and the value of b.

(Total 7 marks)

$\left(1+\tan\frac{1}{2}x\right)^2 = 1+2\tan\left(\frac{1}{2}x\right)+\tan^2\left(\frac{1}{2}x\right)$	M1	Attempt to multiply 3 terms at least 2 correct
$= \sec^2\left(\frac{1}{2}x\right)+2\tan\left(\frac{1}{2}x\right)$	M1	Use of $\sec^2\alpha = 1+\tan^2\alpha$
$\int\left(\sec^2\left(\frac{1}{2}x\right)+2\tan\left(\frac{1}{2}x\right)\right)dx = 2\tan\left(\frac{1}{2}x\right)+2\ln\left(\sec\frac{1}{2}x\right)\times 2$	M1 A1	M1 for attempt to integrate ($k\tan\theta$ or $k\ln\sec\theta$) A1 for all correct
$\int_0^{\frac{\pi}{2}}(\ldots)\,dx = 2\tan\frac{\pi}{4}+4\ln\sec\frac{\pi}{4}-(0)$	M1	Use of limits $\frac{\pi}{4}$ seen (provided some int. attempt)
$= 2+4\ln\sqrt{2}$ $\underline{\mathbf{= 2 + \ln 4}}$	A1 A1 (7)	$a = 2$ $b = 4$ (Accept 2ln2) A1A1 dep. on 4th M only

Question 33 (AEA 2011 Q4)

The curve C has parametric equations

$$x = \cos^2 t$$

$$y = \cos t \sin t$$

where $0 \leqslant t < \pi$

(a) Show that C is a circle and find its centre and its radius.

(5)

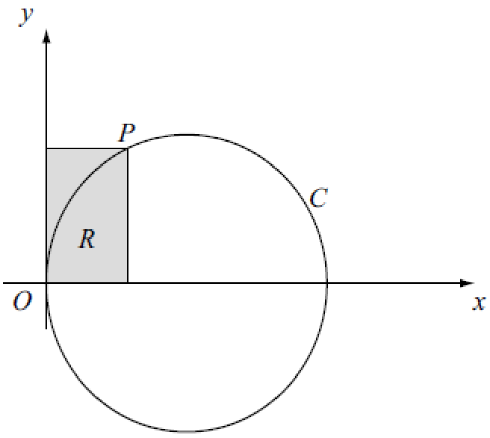


Figure 1

Figure 1 shows a sketch of C. The point P, with coordinates $(\cos^2\alpha\,,\ \cos\alpha\sin\alpha)$, $0 < \alpha < \frac{\pi}{2}$, lies on C. The rectangle R has one side on the x-axis, one side on the y-axis and OP as a diagonal, where O is the origin.

(b) Show that the area of R is $\sin\alpha\,\cos^3\alpha$

(c) Find the maximum area of R, as α varies.

(7)

(a)	$2y = 2\sin t\cos t = \sin 2t$	M1	Use of sin2*t*
	$2x = 2\cos^2 t \ \Rightarrow 2x - 1 = 2\cos^2 t - 1 = \cos 2t$	M1 M1	Use of cos2*t* Successfully eliminating *t* and eqn. for circle
	$(2x-1)^2 + (2y)^2 = 1$		
	$\left(x - \frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$ so centre $\left(\frac{1}{2}, 0\right)$, $r = \frac{1}{2}$	A1A1 (5)	A1 for centre A1 for radius
(b)	Area of $R = \cos^2\alpha \times \sin\alpha\cos\alpha = \cos^3\alpha\sin\alpha$	B1(1)	Some evidence of *xy* leading to given result
(c)	$\frac{\mathrm{d}A}{\mathrm{d}\alpha} = \cos\alpha\cos^3\alpha - 3\cos^2\alpha\sin^2\alpha$	M1A1	M1 for use of product rule
	$\frac{\mathrm{d}A}{\mathrm{d}\alpha} = 0 \Rightarrow \cos^2\alpha\left(\cos^2\alpha - 3\sin^2\alpha\right) = 0$	M1	M1 for setting derivative =0 and attempting to solve
	$\cos^2\alpha = 0 \Rightarrow \left[\alpha = \frac{\pi}{2}\right]$ or $\tan^2\alpha = \frac{1}{3} \Rightarrow \alpha = \frac{\pi}{6}$ (or $30°$)	A1 A1	A1 for "trig" =..A1 for α = .. Can ignore $\alpha = \frac{\pi}{2}$ but consider for S+
	$A'' = 2\sin\alpha\cos\alpha(3 - 8\cos^2\alpha)$ and show <0 for $\alpha = \frac{\pi}{6}$ or argument based on $\alpha = \frac{\pi}{2}$ gives min so this is max	M1	Some check that this value of α gives a max
	Maximum area is $\frac{3\sqrt{3}}{16}$ (o.e.)	B1 (7) **(13)**	Single fraction with rational denom

Question 34 (AEA 2011 Q5)

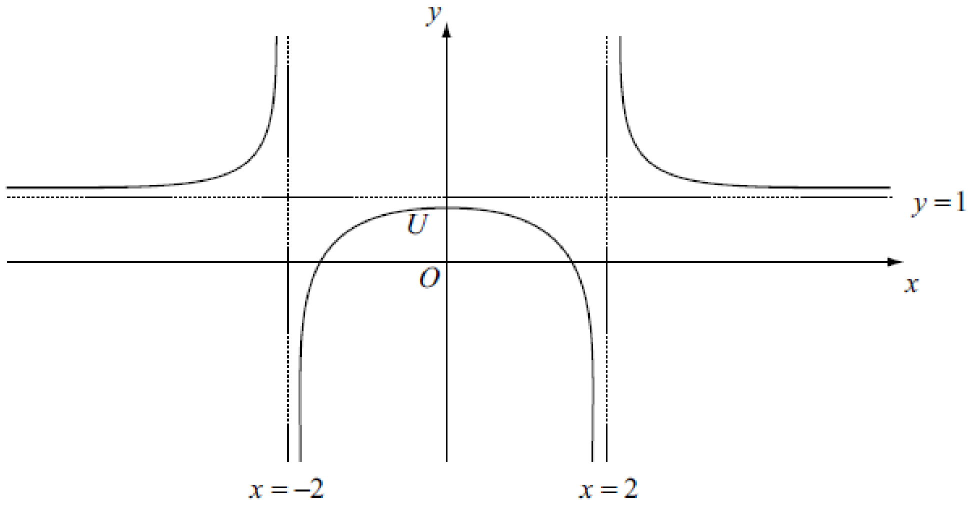


Figure 2

Figure 2 shows a sketch of the curve C with equation $y=\frac{x^2-2}{x^2-4}$ and $x \neq \pm 2$.

The curve cuts the y-axis at U.

(a) Write down the coordinates of the point U.

(1)

The point P with x-coordinate a $(a \neq 0)$ lies on C.

(b) Show that the normal to C at P cuts the y-axis at the point

$$\left(0\ ,\ \left[\frac{a^2-2}{a^2-4}-\frac{\left(a^2-4\right)^2}{4}\right]\right)$$

The circle E, with centre on the y-axis, touches all three branches of C.

(c) (i) Show that

$$\left[\frac{a^2}{2(a^2-4)}-\frac{(a^2-4)^2}{4}\right]^2=a^2+\frac{(a^2-4)^4}{16}$$

(ii) Hence, show that

$$\left(a^2-4\right)^2=1$$

(iii) Find the centre and radius of E.

(10)

(a)	U is $\left(0,\ \tfrac{1}{2}\right)$	B1 (1)	For *y* coordinate
(b)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(x^2-4\right)2x-\left(x^2-2\right)2x}{\left(x^2-4\right)^2},=\frac{-4x}{\left(x^2-4\right)^2}$	M1, A1	M1 for attempt to diff. (Two parts and one correct) Wrong formula used is M0 A1 when num. simplified
	Gradient of normal at P = $\frac{\left(a^2-4\right)^2}{4a}$	M1	Use of perpendicular gradient rule and $x = a$
	Equation of normal: $y-\frac{a^2-2}{a^2-4}=\frac{\left(a^2-4\right)^2}{4a}\left(x-a\right)$	M1	Attempt at eqn of normal can ft their changed grad
	x = 0 gives $y=\frac{a^2-2}{a^2-4}-\frac{\left(a^2-4\right)^2}{4}$ (*)	M1 A1cso (6)	M1 clear use of $x = 0$ in norm A1 for no incorrect working seen

(c)(i)	**No use of circle is 0/5 for (i)**		
	Centre is at (0, k) [where k is y-coord from part (b)]	B1	
	Radius = y coord of their centre – 0.5	B1	May be implied by a sketch radius touches at *U*
	Radius to P = $\sqrt{a^2+\left(k-\frac{a^2-2}{a^2-4}\right)^2}$ or $\sqrt{a^2+\frac{\left(a^2-4\right)^4}{16}}$	M1	Expression for radius from centre to *P*
	From (b) and k - 0.5: $\left[\frac{a^2-2}{a^2-4}-\frac{1}{2}-\frac{\left(a^2-4\right)^2}{4}\right]^2=a^2+\frac{\left(a^2-4\right)^4}{16}$	M1	For attempt at a suitable equation in *a* NB r^2 = LHS implies B1B1
	$\left[\frac{a^2}{2(a^2-4)}-\frac{(a^2-4)^2}{4}\right]^2=a^2+\frac{(a^2-4)^4}{16}$ (*)	A1cso (5)	
(ii)	$\frac{a^4}{4(a^2-4)^2}-\frac{a^2(a^2-4)}{4}+\cancel{\frac{(a^2-4)^4}{16}}=a^2+\cancel{\frac{(a^2-4)^4}{16}}$		[When cancel a^2 and consider $a = 0$ for S+]
	$\frac{a^2}{4(a^2-4)^2}=1+\frac{a^2-4}{4}$ $\left\{=\frac{4+a^2-4}{4}\right\}$	M1	Remove $\frac{\left(a^2-4\right)^2}{16}$ and cancel a^2
	$\left(a^2-4\right)^2=1$ (*)	A1cso	
(iii)	$a^2-4=\pm1$ so $a=\pm\sqrt{3}$ or $\pm\sqrt{5}$	A1	For $a^2=5$ or better, $\sqrt{3}$ can be ignored and $\pm$ Dependent on 3rd M1 [S+ for reason to reject $\sqrt{3}$]
	$k=\frac{5-2}{1}-\frac{1^2}{4}=\frac{11}{4}$ so centre is $\underline{(0,\tfrac{11}{4})\text{ rad is }\tfrac{9}{4}}$	A1A1 (5) **(17)**	A1 for centre, A1 for radius (Dependent on 3rd M1) [May imply some Bs]

Question 35 (AEA 2010 Q5)

$$I = \int \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x, \qquad x > 1$$

(a) Use the substitution $x = 1 + u^{-1}$ to show that

$$I = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c.$$

(7)

(b) Hence show that

$$\int_{\sec\alpha}^{\sec\beta} \frac{1}{(x-1)\sqrt{(x^2-1)}}\,\mathrm{d}x = \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right), \qquad 0 < \alpha < \beta < \tfrac{\pi}{2}$$

(5)

(a)	$x = 1 + u^{-1} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}u} = -\frac{1}{u^2}$	B1	Correct d*x*/d*u* (o.e.)
	$\therefore I = \int \frac{1}{u^{-1}\sqrt{u^{-2}+2u^{-1}}}.\left(-\frac{1}{u^2}\right)\mathrm{d}u$	M1	Attempt to get *I* in *u* only
	$I = -\int \frac{\mathrm{d}u}{\sqrt{1+2u}}$ (o.e)	A1	Correct simplified expression in *u* only
	$= -(1+2u)^{\frac{1}{2}}\,(+c)$	M1 A1	Attempt to int' their *I* Correct integration
	Uses $u = \frac{1}{x-1}$ to give $I = -(1+\frac{2}{x-1})^{\frac{1}{2}} + c$, $I = -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}} + c$	M1 A1cso (7)	Sub back in *x*s Including + *c*
(b)	$= -\left(\frac{\sec\beta+1}{\sec\beta-1}\right)^{\frac{1}{2}} + \left(\frac{\sec\alpha+1}{\sec\alpha-1}\right)^{\frac{1}{2}}$	M1	Use of part (a)
	$= -\left(\frac{1+\cos\beta}{1-\cos\beta}\right)^{\frac{1}{2}} + \left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)^{\frac{1}{2}}$	M1	Multiply by cos*x*
	$= -\left(\frac{2\cos^2(\frac{\beta}{2})}{2\sin^2(\frac{\beta}{2})}\right)^{\frac{1}{2}} + \left(\frac{2\cos^2(\frac{\alpha}{2})}{2\sin^2(\frac{\alpha}{2})}\right)^{\frac{1}{2}}$ ["2" is needed]	M1	Use of half angle formulae
	$= \cot\left(\frac{\alpha}{2}\right) - \cot\left(\frac{\beta}{2}\right)$ (*)	M1 A1cso (5) **[12]**	Correct removal of $\sqrt{\ }$.

Question 36 (AEA 2010 Q7)

$$f(x) = [1+\cos(x+\tfrac{\pi}{4})][1+\sin(x+\tfrac{\pi}{4})], \qquad 0 \leqslant x \leqslant 2\pi$$

(a) Show that $f(x)$ may be written in the form

$$f(x) = (\tfrac{1}{\sqrt{2}} + \cos x)^2, \qquad 0 \leqslant x \leqslant 2\pi$$

(5)

(b) Find the range of the function $f(x)$.

(2)

The graph of $y = f(x)$ is shown in Figure 2.

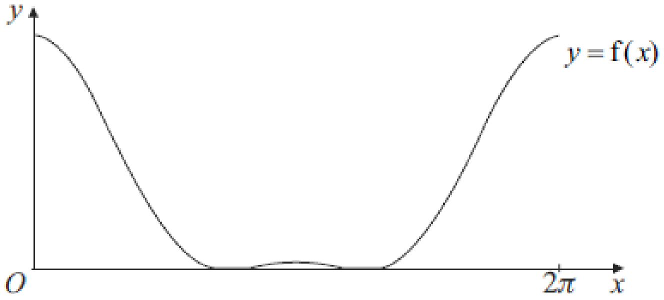


Figure 2

(c) Find the coordinates of all the maximum and minimum points on this curve.

(6)

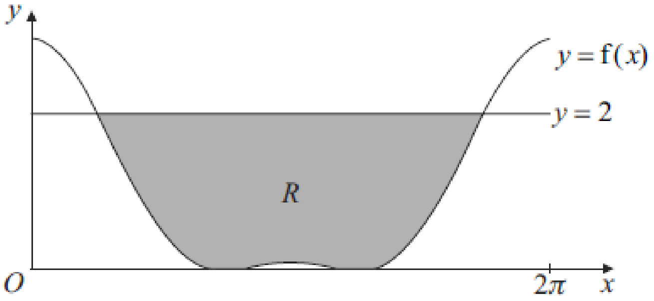


Figure 3

The region R, bounded by $y = 2$ and $y = f(x)$, is shown shaded in Figure 3.

(d) Find the area of R.

(8)

7(a)	$\mathrm{f}(x)=[1+(\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4})][1+(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4})]$	M1	Use of $\sin(A\pm B)$ etc
	$=[1+\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x)][1+\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x]$	B1	$\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$
	$=(1+\frac{1}{\sqrt{2}}\cos x)^2-(\frac{1}{\sqrt{2}}\sin x)^2$ **or** $=1+\frac{2}{\sqrt{2}}\cos x+\frac{1}{2}\cos^2 x-\frac{1}{2}\sin^2 x$	M1	Multiply out and remove sin*x*cos*x* terms
	$=1+\frac{2}{\sqrt{2}}\cos x+\frac{1}{2}\cos^2 x-\frac{1}{2}(1-\cos^2 x)$		
	So f(x) $=\frac{1}{2}+\frac{2}{\sqrt{2}}\cos x+\cos^2 x \quad =(\frac{1}{\sqrt{2}}+\cos x)^2$ (*)	M1 A1cso (5)	Eqn in cos*x* only
(b)	Range: $0\le\mathrm{f}(x)\le(\frac{1}{\sqrt{2}}+1)^2$ or equivalent e.g. $\frac{3}{2}+\frac{2}{\sqrt{2}}$	M1 A1 (2)	M1 $\mathrm{f}\underline{>}0$ <u>or</u> $\mathrm{f}\underline{<}(\frac{1}{\sqrt{2}}+1)^2$ A1 both [M1A0 for <]
(c)	$\cos x=1$ gives maxima at $(0,\frac{3}{2}+\sqrt{2})$ and at $(2\pi,\frac{3}{2}+\sqrt{2})$	B1 B1ft	If *y* co-ord is wrong allow 2nd B1ft
	Minima when $(\frac{1}{\sqrt{2}}+\cos x)=0\Rightarrow\cos x=-\frac{1}{\sqrt{2}}$ so at $x=\frac{3\pi}{4}$ or $\frac{5\pi}{4}$	M1 A1	M1 for $y=0$ at cos*x* = A1 for *x* co-ords
	$\mathrm{f}'(x)=-2\sin x(\frac{1}{\sqrt{2}}+\cos x)=0$ at $x=\pi$, so at $(\pi,\frac{3}{2}-\sqrt{2})$ there is a (local) maximum	M1 A1 (6)	For f '(x)=0 and $x=\pi$ A1for max point
(d)	$y=2$ meets $y=\mathrm{f}(x)$ so $(\frac{1}{\sqrt{2}}+\cos x)^2=2\Rightarrow\cos x=\frac{\sqrt{2}}{2}$ $\therefore x=\frac{\pi}{4}$ or $\frac{7\pi}{4}$	M1 A1	Form and solve correct eqn Both
	Area = $\int(2-\mathrm{f}(x))\,\mathrm{d}x$ [or correct rect - integral o.e.]	M1	Correct strategy
	$=\int\left(1-\sqrt{2}\cos x-\frac{1}{2}\cos 2x\right)\mathrm{d}x$	M1	All terms of integral in suitable form
	$=\left[x-\sqrt{2}\sin x-\frac{1}{4}\sin 2x\right]$	dM1A1	M1 for some correct int' Dep on previous M A1 for all correct
	$=\left(\frac{7\pi}{4}+\sqrt{2}\times\frac{1}{\sqrt{2}}+\frac{1}{4}\times 1\right)-\left(\frac{\pi}{4}-\sqrt{2}\times\frac{1}{\sqrt{2}}-\frac{1}{4}\right)$	dM1	Use of their correct limits. Dep on 1st M1
	$=\underline{\frac{3\pi}{2}+\frac{5}{2}}$	A1 (8) **[21]**	NB Rectangle $=3\pi$
ALT	(a) $\mathrm{f}(x)=1+\sqrt{2}\cos(x+\frac{\pi}{4}-\frac{\pi}{4})+\frac{1}{2}\sin(2x+\frac{\pi}{2})$	1stM1B1	
	$=1+\sqrt{2}\cos x+\frac{1}{2}\cos 2x$	2nd M1	Remove $\sin(2x+\frac{\pi}{2})$
	$=1+\sqrt{2}\cos x-\frac{1}{2}+\cos^2 x$	3rd M1	Then as in scheme
ALT	(d) $\int(\frac{1}{\sqrt{2}}+\cos x)^2\mathrm{d}x=\int\frac{1}{2}+\sqrt{2}\cos x+\frac{1}{2}+\frac{1}{2}\cos 2x\ \mathrm{d}x$	3rd M1	All terms in form to int'
	$=\frac{1}{2}x+\sqrt{2}\sin x+\frac{1}{4}\sin 2x+\frac{1}{2}x$	4th M1 2nd A1	Will score 2nd M1 when they try to subtract from area of rectangle

Question 37 (AEA 2009 Q6)

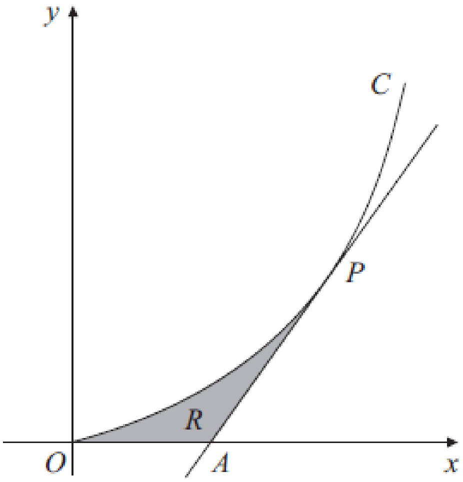


Figure 2

Figure 2 shows a sketch of the curve C with parametric equations

$$x = 2\sin t, \qquad y = \ln(\sec t), \qquad 0 \leqslant t < \frac{\pi}{2}.$$

The tangent to C at the point P, where $t = \frac{\pi}{3}$, cuts the x-axis at A.

(a) Show that the x-coordinate of A is $\frac{\sqrt{3}}{3}(3 - \ln 2)$.

(6)

The shaded region R lies between C, the positive x-axis and the tangent AP as shown in Figure 2.

(b) Show that the area of R is $\sqrt{3}(1 + \ln 2) - 2\ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{6}(\ln 2)^2$.

(11)

(a)	P is $(\sqrt{3}, \ln 2)$	B1	Score anywhere.
	$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\tan t}{2\cos t}$	M1 A1	M1 attempt $\frac{dy}{dx}$ A1 correct
	When $t = \frac{\pi}{3}$ $\quad m = \sqrt{3}$	A1	
	Equation of tangent at P is: $\quad y - \ln 2 = \sqrt{3}(x - \sqrt{3})$	M1	Attempt tangent at P. √ their P and m
	A is where $y = 0 \quad \therefore \quad -\frac{\ln 2}{\sqrt{3}} + \sqrt{3} = x \Rightarrow (x =)\frac{\sqrt{3}}{3}(3 - \ln 2)$	A1 cso (6)	Allow $\frac{3 - \ln 2}{\sqrt{3}}$
(b)	Area under curve $= \int_{t=0}^{t=\pi/3} y\,\mathrm{d}x = \int_{(0)}^{(\pi/3)} \ln\sec t . 2\cos t\,\mathrm{d}t$	M1	Attempt $\int y\dot{x}\mathrm{d}t$ √ $\dot{x}$ condone missing 2
	$= [2\sin t \ln\sec t] - \int 2\sin t \tan t \; dt$	M1 A1	Attempt parts. Both parts correct.
	$= [\qquad\qquad] - \int 2\frac{(1-\cos^2 t)}{\cos t}dt$	M1	Use of $s^2 = 1 - c^2$
	$= [\qquad\qquad] - 2\int \sec t \; dt + 2\int \cos t \; dt$	M1	Split
	$= [2\sin t \ln\sec t] - \underline{2\ln\lvert\sec t + \tan t\rvert} \; \underline{\underline{+ 2\sin t}}$	A1, A1	Accept $\underline{\cos t \tan t}$
	$= \sqrt{3}\ln 2 \qquad - (2\ln[2 + \sqrt{3}] - 0) + (2\frac{\sqrt{3}}{2} - 0)$	M1	Use of correct limits on all 3 integrals
	$\left[= \sqrt{3}(\ln 2 + 1) - 2\ln\left(2 + \sqrt{3}\right)\right]$		
	Area of $\Delta \quad = \frac{1}{2}\left[\sqrt{3} - \frac{\sqrt{3}}{3}(3 - \ln 2)\right]\ln 2 \quad \left\{= \frac{\sqrt{3}}{6}(\ln 2)^2\right\}$	B1	Any correct expression.
	Area of $R \quad$ = are under curve – area of Δ	M1	Strategy must be $\int$ or area
	$= \sqrt{3}(\ln 2 + 1) - 2\ln(2 + \sqrt{3}) - \frac{\sqrt{3}}{6}(\ln 2)^2 \quad (*)$	A1 cso (11) [17]	

Question 38 (AEA 2008 Q2)

The points (x, y) on the curve C satisfy

$$(x+1)(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} = xy.$$

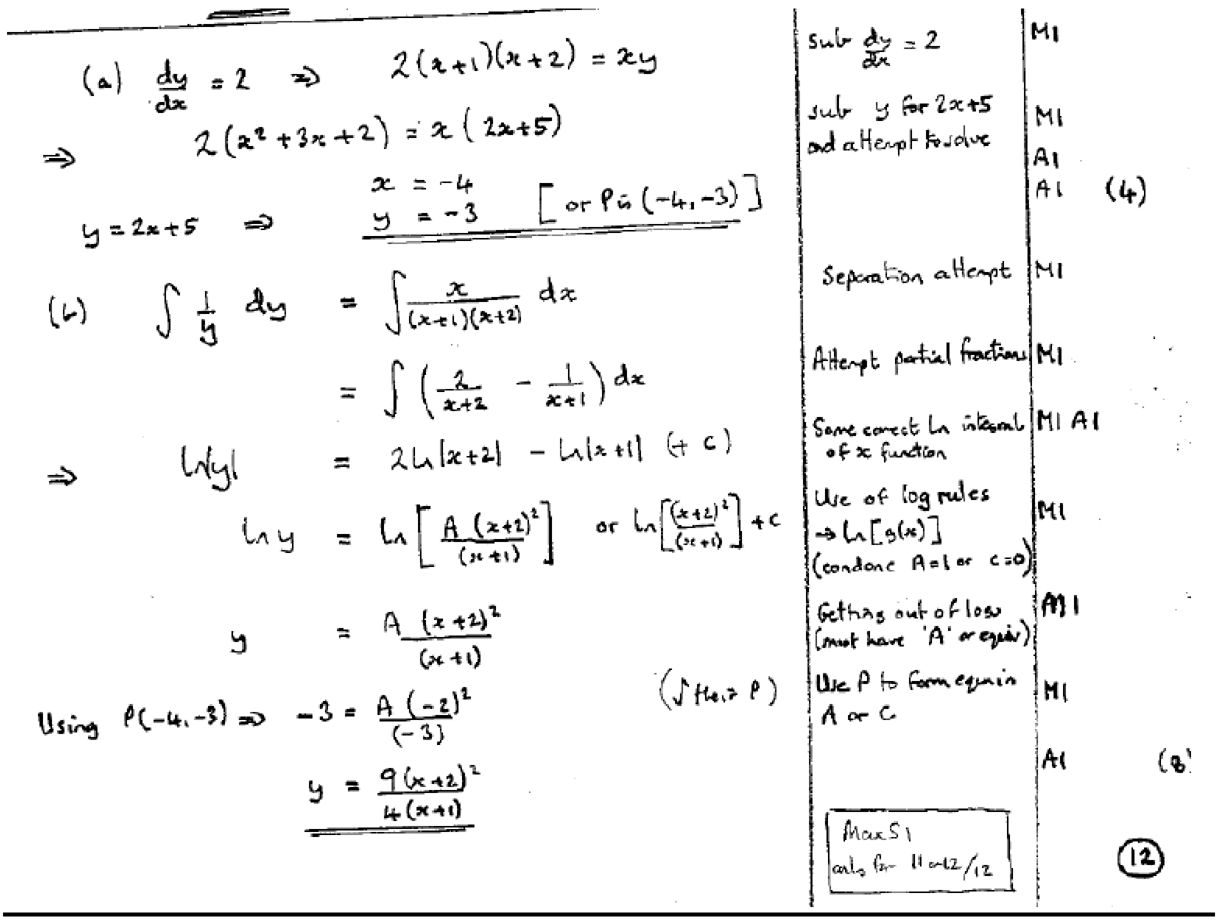
The line with equation $y = 2x + 5$ is the tangent to C at a point P.

(a) Find the coordinates of P.

(4)

(b) Find the equation of C, giving your answer in the form $y = \mathrm{f}(x)$.

(8)



(a) $\frac{dy}{dx} = 2 \Rightarrow 2(x+1)(x+2) = xy$	Sub $\frac{dy}{dx} = 2$	M1	
$\Rightarrow 2(x^2+3x+2) = x(2x+5)$	sub y for $2x+5$ and attempt to solve	M1	
$x = -4$		A1	
$y = 2x+5 \Rightarrow y = -3$ [or P is $(-4, -3)$]		A1	(4)
(b) $\int \frac{1}{y}\,dy = \int \frac{x}{(x+1)(x+2)}\,dx$	Separation attempt	M1	
$= \int \left(\frac{2}{x+2} - \frac{1}{x+1}\right) dx$	Attempt partial fractions	M1	
$\Rightarrow \ln\lvert y\rvert = 2\ln\lvert x+2\rvert - \ln\lvert x+1\rvert\ (+c)$	Some correct ln integral of x function	M1 A1	
$\ln y = \ln\left[\frac{A(x+2)^2}{(x+1)}\right]$ or $\ln\left[\frac{(x+2)^2}{(x+1)}\right] + c$	Use of log rules $\rightarrow \ln[g(x)]$ (condone $A=1$ or $c=0$)	M1	
$y = \frac{A(x+2)^2}{(x+1)}$	Getting out of logs (must have 'A' or equiv)	M1	
Using $P(-4, -3) \Rightarrow -3 = \frac{A(-2)^2}{(-3)}$ ($\sqrt{}$ their P)	Use P to form eqn in A or C	M1	
$y = \frac{9(x+2)^2}{4(x+1)}$		A1	(8)
	Max S1 only for 11 or 12/12		⑫

Question 39 (AEA 2008 Q4)

Figure 1

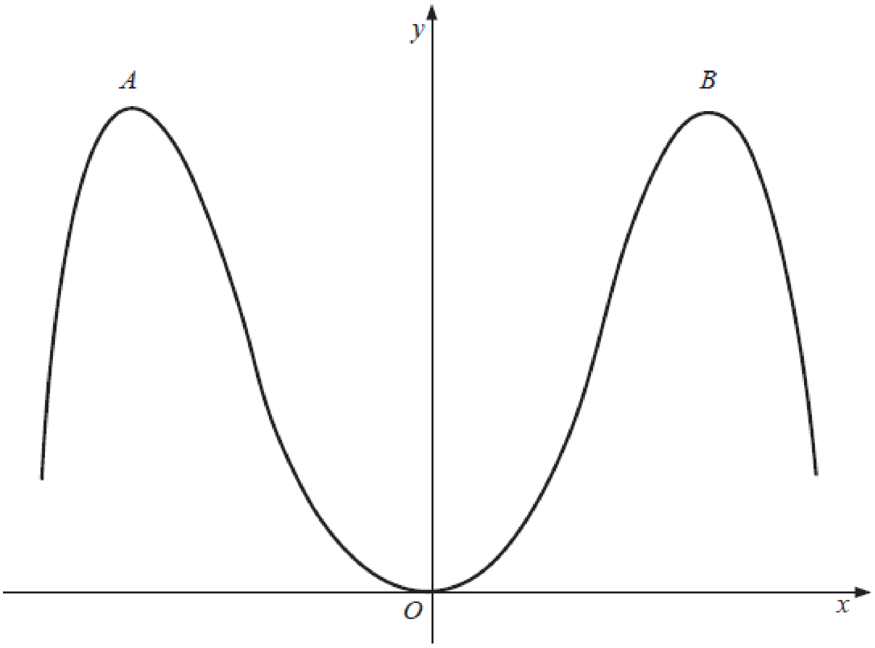


Figure 1 shows a sketch of the curve C with equation

$$y = \cos x \ln(\sec x), \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

The points A and B are maximum points on C.

(a) Find the coordinates of B in terms of e.

(5)

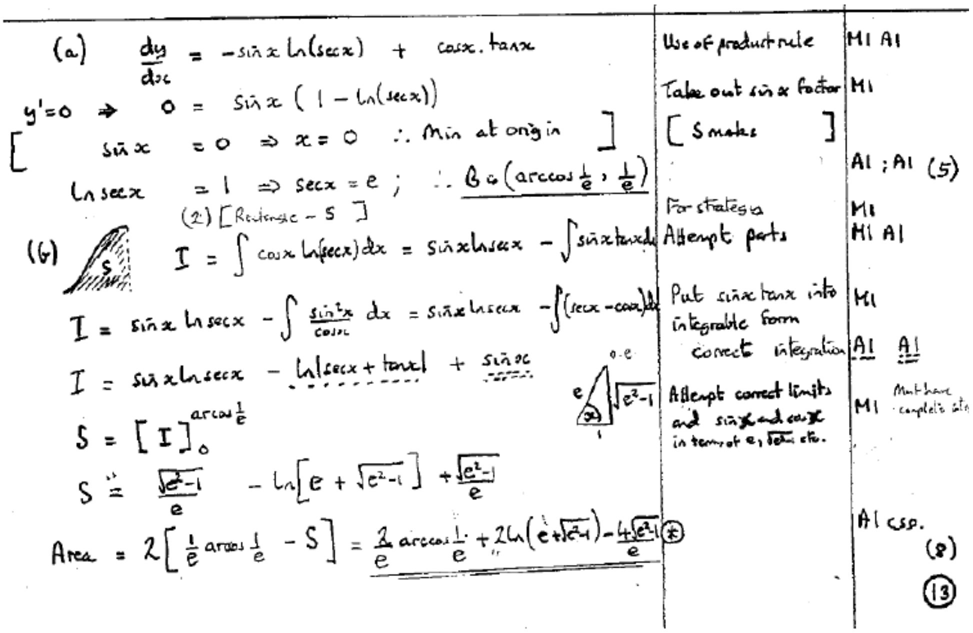
The finite region R lies between C and the line AB.

(b) Show that the area of R is

$$\frac{2}{\mathrm{e}}\arccos\left(\frac{1}{\mathrm{e}}\right) + 2\ln\left(\mathrm{e} + \sqrt{(\mathrm{e}^2 - 1)}\right) - \frac{4}{\mathrm{e}}\sqrt{(\mathrm{e}^2 - 1)}.$$

[arccos$x$ is an alternative notation for $\cos^{-1}x$]

(8)



(a) $\frac{dy}{dx} = -\sin x \ln(\sec x) + \cos x . \tan x$	Use of product rule	M1 A1
$y'=0 \Rightarrow 0 = \sin x\,(1 - \ln(\sec x))$	Take out sin x factor	M1
[$\sin x = 0 \Rightarrow x = 0$ ∴ Min at origin]	[S marks]	
$\ln \sec x = 1 \Rightarrow \sec x = e$; ∴ B is $\left(\arccos\frac{1}{e}, \frac{1}{e}\right)$		A1 ; A1 (5)
(2) [Rectangle − S]	For strategy	M1
(b) S $I = \int \cos x \ln(\sec x)dx = \sin x \ln \sec x - \int \sin x \tan x\,dx$	Attempt parts	M1 A1
$I = \sin x \ln \sec x - \int \frac{\sin^2 x}{\cos x}dx = \sin x \ln \sec x - \int(\sec x - \cos x)dx$	Put sin x tan x into integrable form	M1
$I = \sin x \ln \sec x - \ln\lvert\sec x + \tan x\rvert + \sin x$	correct integration	A1 A1
0·e e, $\sqrt{e^2-1}$, x, 1 $S = [I]_0^{\arccos\frac{1}{e}}$	Attempt correct limits and sin x and cos x in terms of e, $\sqrt{e^2-1}$ etc.	M1 Must have complete int
$S \doteq \frac{\sqrt{e^2-1}}{e} - \ln\left[e + \sqrt{e^2-1}\right] + \frac{\sqrt{e^2-1}}{e}$		
Area $= 2\left[\frac{1}{e}\arccos\frac{1}{e} - S\right] = \frac{2}{e}\arccos\frac{1}{e} + 2\ln\left(e + \sqrt{e^2-1}\right) - \frac{4\sqrt{e^2-1}}{e}$ ⊛		A1 cso. (8)
		⑬

Question 40 (AEA 2007 Q4)

The function h(x) has domain $\mathbb{R}$ and range h(x) > 0, and satisfies

$$\sqrt{\int \mathrm{h}(x)\ \mathrm{d}x} = \int \sqrt{\mathrm{h}(x)}\ \mathrm{d}x.$$

(*a*) By substituting $\mathrm{h}(x) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2(y + c),$$

where c is constant.

(5)

(*b*) Hence find a general expression for y in terms of x.

(4)

(*c*) Given that h(0) = 1, find h(x).

(2)

$\left(\sqrt{\int h(x)dx} =\right) \int \frac{dy}{dx}\, dx = y\ (+c)$	Sub and $\int$ in RHS condone missing $+c$	M1
Square $\int h(x)\, dx = (y+c)^2$	squaring	M1
Differentiate $h(x) = \left(\frac{dy}{dx}\right)^2 = 2(y+c).\left(\frac{dy}{dx}\right)$	differentiate	M1
$\Rightarrow \frac{dy}{dx} = 2(y+c)$ ⊛ $\left[\because h>0 \therefore \frac{dy}{dx} \neq 0\right]$	$\div$ by $\frac{dy}{dx}$	M1 A1 cso
	[for 5 marks]	(5)
(b) $\int \frac{1}{y+c}\, dy = 2\int dx$	Separation	M1
$\ln\|y+c\| = 2x\ (+\ \alpha)$	correct - condone missing constant	A1
$y+c = Ae^{2x}$	out of logs	M1
$\underline{y = Ae^{2x} - c}$ or $Ae^{2x} + k$	A and $\pm c$ needed	A1 (4)
(c) $\frac{dy}{dx} = 2Ae^{2x}$		
$\therefore\ h(x) = \left(\frac{dy}{dx}\right)^2 = 4A^2e^{4x}$	An expression for h with arbitrary const.	M1
$h(0) = 1 \Rightarrow 4A^2 = 1 \quad \therefore\ \underline{h(x) = e^{4x}}$	c.s.o.	A1 (2) ⑪

Question 41 (AEA 2006 Q6)

Figure 1

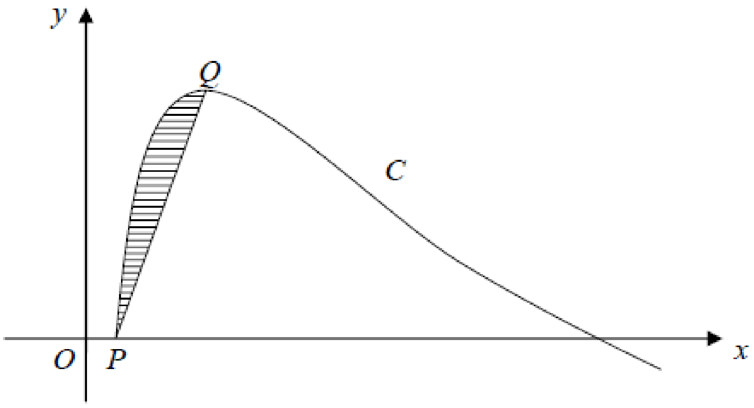


Figure 1 shows a sketch of part of the curve C with equation

$$y = \sin(\ln x), \quad x \geq 1.$$

The point Q, on C, is a maximum.

(*a*) Show that the point $P(1, 0)$ lies on C.

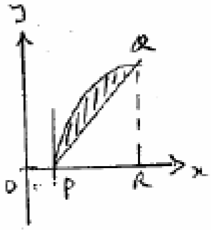
(1)

(*b*) Find the coordinates of the point Q.

(5)

(*c*) Find the area of the shaded region between C and the line PQ.

(9)

1	(a) $x=1$; $y=\sin(\ln 1)=\sin 0=0$	
	$\therefore$ $P=(1,0)$ and P lies on C	B1 c.s.o (1
(b)	$y'=\frac{1}{x}\cos(\ln x)$	M1, A1
	$y'=0$ at Q $\therefore \cos(\ln x)=0$ $\therefore \ln x=\pi/2$, $x=e^{\pi/2}$	M1
	$\therefore Q=(e^{\pi/2}, \sin(\ln e^{\pi/2}))$	A1
	$=\underline{(e^{\pi/2}, 1)}$	A1 (5
(c)	 $\text{Area}=\int_1^{e^{\pi/2}}\sin(\ln x)\,dx-\text{Area }\Delta PQR$	M1 (correct approach)
	$\text{Area }\Delta PQR=\frac{1}{2}\times 1\times(e^{\pi/2}-1)$	B1
	For integral; let $\ln x=u$ $\therefore x=e^{u}$ $\frac{1}{x}dx=du$ $\therefore dx=e^{u}du$	(subst) M1
	$I=\int_0^{\pi/2}\sin u\,(e^{u}du)$	($\int$ in u) A1
	$=[e^{u}\sin u]_0^{\pi/2}-\int e^{u}\cos u\,du$	(limits) M1 (parts) M1
	$=e^{\pi/2}-[e^{u}\cos u]_0^{\pi/2}-\int e^{u}\sin u\,du$	(second parts) M1
	$\therefore 2I=e^{\pi/2}+1$ $I=\frac{1}{2}(1+e^{\pi/2})$	(I) A1
	$\therefore \text{Area}=\frac{1}{2}(1+e^{\pi/2})-\frac{1}{2}(-1+e^{\pi/2})=\underline{1}$	A1 (9)

Question 42 (AEA 2005 Q3)

Given that

$$\frac{\mathrm{d}}{\mathrm{d}x}(u\sqrt{x}) = \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}(\sqrt{x})}{\mathrm{d}x}, \quad 0 < x < \tfrac{1}{2},$$

where u is a function of x, and that $u = 4$ when $x = \frac{3}{8}$, find u in terms of x.

(9)

$\frac{d}{dx}(u\sqrt{x}) = \sqrt{x}\frac{du}{dx} + \frac{1}{2\sqrt{x}}\cdot u$ (product rule)	M1
$\therefore$ Eqn $\Rightarrow$ $\sqrt{x}\frac{du}{dx} + \frac{1}{2\sqrt{x}}\cdot u = \frac{1}{2\sqrt{x}}\frac{du}{dx}$ (D.E.) (all $\frac{d}{dx}$'s completed)	A1
$\therefore \frac{du}{dx}\left(\frac{1}{2\sqrt{x}} - \sqrt{x}\right) = \frac{1}{2\sqrt{x}}u$	
$\therefore \frac{du}{dx}(1-2x) = u$ (simplification)	M1, A1
$\therefore \int \frac{du}{u} = \int \frac{dx}{1-2x}$ (sep^n of variables)	M1
$\therefore \ln u = -\frac{1}{2}\ln(1-2x)\ \left[+\frac{1}{2}\ln k\right]$ (int.^{n}) (or $\ln(2x-1)$)	M1 (2 int terms) A1 (c a o)
~~Since $0 < x < \frac{1}{2}$, $\ln u = -\frac{1}{2}\ln(1-2x) + \frac{1}{2}\ln k$~~	
~~$u^2 = \frac{k}{1-2x}$~~	
$u = 4,\ x = 3/8 \Rightarrow 16 = \frac{k}{1-3/4} \Rightarrow k = 4$ (use of condition)	M1
$\therefore u = 2(1-2x)^{-1/2}$	A1
	(9)

Question 43 (AEA 2005 Q7)

(*a*) Use the substitution $x = \sec\theta$ to show that

$$\int \sqrt{(x^2 - 1)}\ \mathrm{d}x$$

can be written as

$$\int \sec\theta \tan^2\theta\ \mathrm{d}\theta .$$

(3)

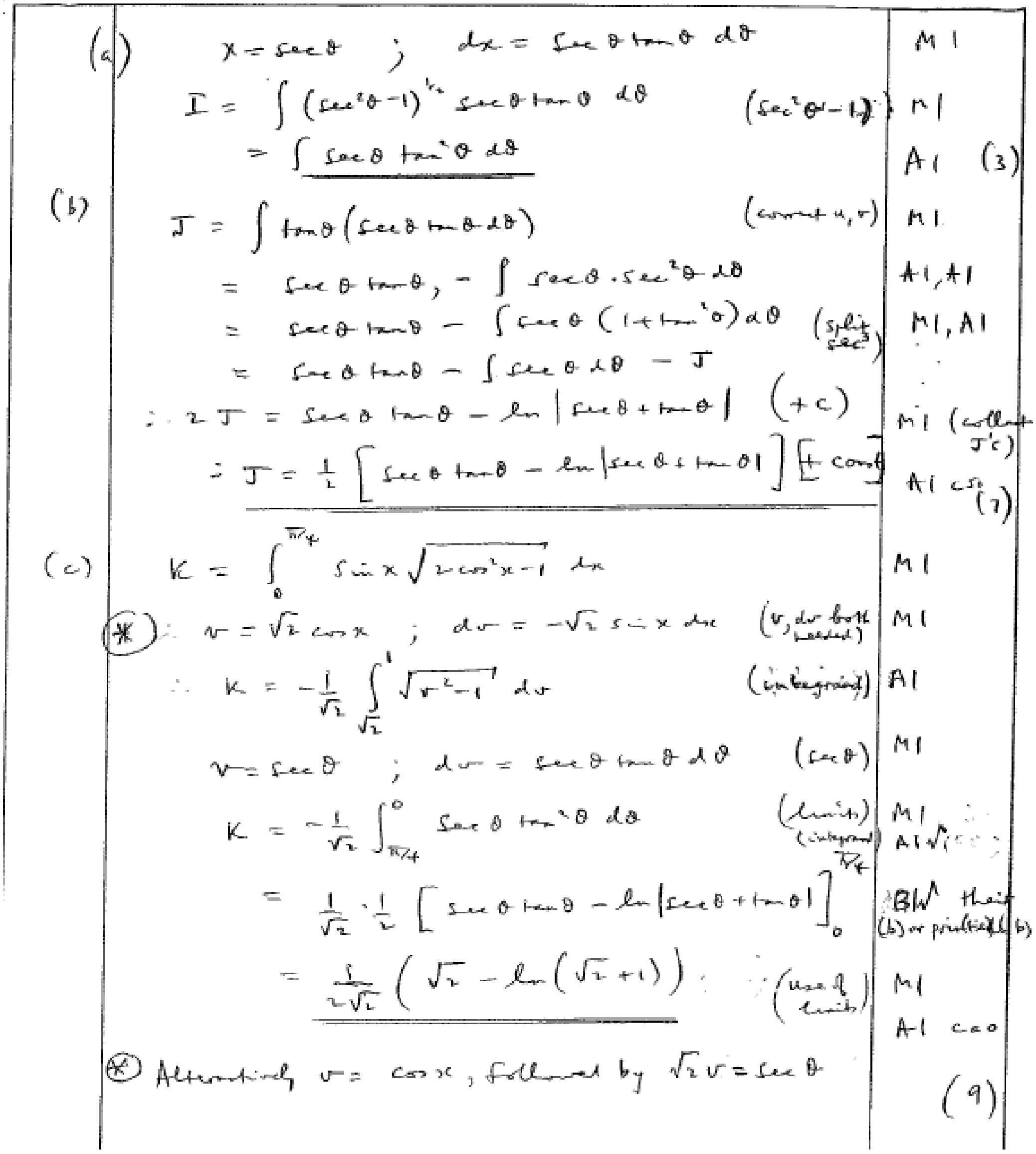
(*b*) Use integration by parts to show that

$$\int \sec\theta \tan^2\theta\ \mathrm{d}\theta = \frac{1}{2}\left[\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\right] + \text{constant}.$$

(7)

(*c*) Evaluate $\int_0^{\frac{\pi}{4}} \sin x\sqrt{(\cos 2x)}\ \mathrm{d}x$.

(9)



(a)	$x = \sec\theta$; $dx = \sec\theta\tan\theta\ d\theta$		M1
	$I = \int (\sec^2\theta - 1)^{1/2} \sec\theta\tan\theta\ d\theta$	$(\sec^2\theta - 1)$	M1
	$\Rightarrow \underline{\int \sec\theta\tan^2\theta\ d\theta}$		A1 (3)
(b)	$J = \int \tan\theta(\sec\theta\tan\theta\, d\theta)$	(correct u, v)	M1
	$= \sec\theta\tan\theta, - \int \sec\theta \cdot \sec^2\theta\, d\theta$		A1, A1
	$= \sec\theta\tan\theta - \int \sec\theta(1+\tan^2\theta)\, d\theta$	(split $\sec^3$)	M1, A1
	$= \sec\theta\tan\theta - \int \sec\theta\, d\theta - J$		
	$\therefore 2J = \sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert$ (+c)		M1 (collect J's)
	$\therefore J = \frac{1}{2}\left[\sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert\right]$ [+ const]		A1 cso (7)
(c)	$K = \int_0^{\pi/4} \sin x\sqrt{2\cos^2 x - 1}\ dx$		M1
	✱ $v = \sqrt{2}\cos x$; $dv = -\sqrt{2}\sin x\, dx$	(v, dv both needed)	M1
	$\therefore K = -\frac{1}{\sqrt{2}}\int_{\sqrt{2}}^{1} \sqrt{v^2 - 1}\ dv$	(integrand)	A1
	$v = \sec\theta$; $dv = \sec\theta\tan\theta\, d\theta$	$(\sec\theta)$	M1
	$K = -\frac{1}{\sqrt{2}}\int_{\pi/4}^{0} \sec\theta\tan^2\theta\ d\theta$	(limits) (integrand)	M1 A1
	$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\left[\sec\theta\tan\theta - \ln\lvert\sec\theta + \tan\theta\rvert\right]_0^{\pi/4}$		B1 ft their (b) or printed (b)
	$= \underline{\frac{1}{2\sqrt{2}}\left(\sqrt{2} - \ln(\sqrt{2}+1)\right)}$	(use of limits)	M1 A1 cao
	✱ Alternatively $v = \cos x$, followed by $\sqrt{2}v = \sec\theta$		(9)

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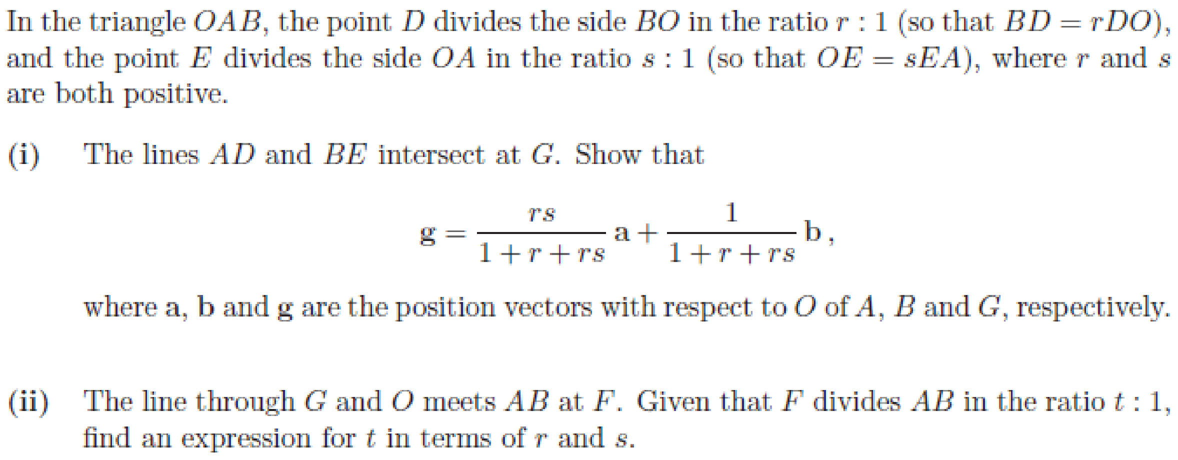
2.12 VECTORS (3D CORDS)

No questions available.

2.13 VECTORS FM (VECTOR EQUATIONS OF LINES – NO LONGER IN STANDARD A LEVEL)

Question 1 (STEP I 2014 Q7)



In the triangle OAB, the point D divides the side BO in the ratio $r:1$ (so that $BD = rDO$), and the point E divides the side OA in the ratio $s:1$ (so that $OE = sEA$), where r and s are both positive.

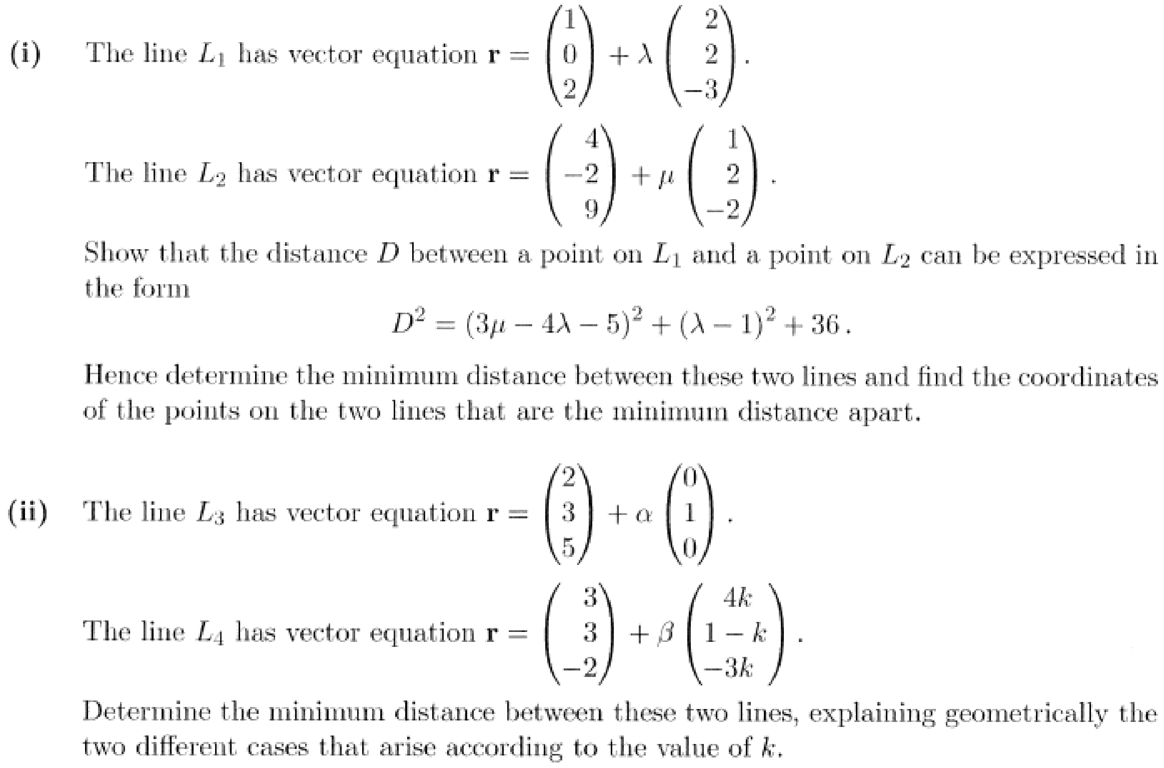
(i) The lines AD and BE intersect at G. Show that

$$\mathbf{g} = \frac{rs}{1+r+rs}\,\mathbf{a} + \frac{1}{1+r+rs}\,\mathbf{b}\,,$$

where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{g}$ are the position vectors with respect to O of A, B and G, respectively.

(ii) The line through G and O meets AB at F. Given that F divides AB in the ratio $t:1$, find an expression for t in terms of r and s.

Question 2 (STEP I 2007 Q7)



(i) The line L_1 has vector equation $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-3\end{pmatrix}$.

The line L_2 has vector equation $\mathbf{r} = \begin{pmatrix}4\\-2\\9\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

Show that the distance D between a point on L_1 and a point on L_2 can be expressed in the form

$$D^2 = (3\mu - 4\lambda - 5)^2 + (\lambda - 1)^2 + 36\,.$$

Hence determine the minimum distance between these two lines and find the coordinates of the points on the two lines that are the minimum distance apart.

(ii) The line L_3 has vector equation $\mathbf{r} = \begin{pmatrix}2\\3\\5\end{pmatrix} + \alpha\begin{pmatrix}0\\1\\0\end{pmatrix}$.

The line L_4 has vector equation $\mathbf{r} = \begin{pmatrix}3\\3\\-2\end{pmatrix} + \beta\begin{pmatrix}4k\\1-k\\-3k\end{pmatrix}$.

Determine the minimum distance between these two lines, explaining geometrically the two different cases that arise according to the value of k.

Solutions: (i) $D = \sqrt{6}$, $(3,2,-1)$, $(7,4,3)$ (ii) $D = \sqrt{50}$ if lines are parallel, $D = 5$ otherwise

Question 3 (AEA 2013 Q3)

The lines L_1 and L_2 have equations given by

$$L_1: \quad \mathbf{r} = \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \quad \text{and} \quad L_2: \quad \mathbf{r} = \begin{pmatrix} 7 \\ p \\ -6 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ -4 \\ -1 \end{pmatrix}$$

where λ and μ are parameters and p is a constant.

The two lines intersect at the point C.

(a) Find

(i) the value of p,

(ii) the position vector of C.

(5)

(b) Show that the point B with position vector $\begin{pmatrix} -13 \\ 11 \\ -4 \end{pmatrix}$ lies on L_2.

(1)

The point A with position vector $\begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix}$ lies on L_1.

(c) Find $\cos(\angle ACB)$, giving your answer as an exact fraction.

(3)

The line L_3 bisects the angle ACB.

(d) Find a vector equation of L_3.

(4)

Question	Scheme	Marks	Notes
(a)	$-7+2\lambda = 7+10\mu$ and $1-3\lambda = -6-\mu$ (o.e.)	M1	Form suitable eqns
	$\Rightarrow 14\mu = -14$ $\quad\underline{\mu = -1,\ (\lambda = 2)}$	M1A1	M1 for eqn in 1 var
	Check in 3rd equation: $7 = p - 4\mu$ $\quad\underline{p = 3}$	A1	Check in 3rd, $p = \ldots$
	Position vector of C is $\begin{pmatrix} -3 \\ 7 \\ -5 \end{pmatrix}$	A1 (5)	Accept as coordinates
(b)	$\mu = -2 \Rightarrow 7 - 2\times 10 = -13,\ 3 - 2\times -4 = 11$ and $-6 - 2\times -1 = -4$	B1 (1)	See $\mu = -2$ & ans
(c)	$\overrightarrow{CA} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$ and $\overrightarrow{CB} = \begin{pmatrix} -10 \\ 4 \\ 1 \end{pmatrix}$ giving $\overrightarrow{CA} \bullet \overrightarrow{CB} = 40 + 0 + 6 = 46$	M1	Attempts a suitable scalar product. Allow 1 sign slip Allow $\underline{+}$
	$\cos(ACB) = \dfrac{46}{\sqrt{52}\sqrt{117}}, = \dfrac{46}{2\sqrt{13}\times 3\sqrt{13}} \quad = \dfrac{23}{39}$ (o.e.)	dM1 A1 (3)	Allow $\underline{+}$ A1 for an exact fraction (no surds)
(d)	Form Rhombus. Let $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA}$ then $\overrightarrow{CD} = \overrightarrow{CB} + 3\overrightarrow{CM}$	M1	Attempt suitable rhombus or unit vectors
	$\overrightarrow{CD} = \begin{pmatrix} -16 \\ 4 \\ 10 \end{pmatrix}$ <u>or</u> $\overrightarrow{OD} = \begin{pmatrix} -19 \\ 11 \\ 5 \end{pmatrix}$	A1	
	$\mathbf{r} = \overrightarrow{OC} + t\overrightarrow{CD}, \quad \mathbf{r} = \begin{pmatrix} -3 \\ 7 \\ -5 \end{pmatrix} + t\begin{pmatrix} -8 \\ 2 \\ 5 \end{pmatrix}$ (o.e.)	dM1 A1 (4) **(13)**	Dep. On 1st M1. For attempt equation of line

Question 4 (AEA 2012 Q4)

$$\mathbf{a} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 8 \\ -4 \\ 3 \end{pmatrix}$$

The points A, B and C with position vectors **a**, **b** and **c**, respectively, are 3 vertices of a cube.

(a) Find the volume of the cube.

(5)

The points P, Q and R are vertices of a second cube with $\overrightarrow{PQ} = \begin{pmatrix} 3 \\ 4 \\ \alpha \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix}$

and α a positive constant.

(b) Given that angle $QPR = 60°$, find the value of α.

(3)

(c) Find the length of a diagonal of the second cube.

(3)

Qu	Scheme	Mark	Notes
(a)	$\overrightarrow{AB} = \begin{pmatrix} 8 \\ -3 \\ 5 \end{pmatrix}$ $\overrightarrow{BC} = \begin{pmatrix} 3 \\ -2 \\ -6 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} 11 \\ -5 \\ -1 \end{pmatrix}$ Only 3 vertex-vertex distances in a cube	M1	Attempt all of these three vectors or two and show perpendicular
	$\lvert AB\rvert = \sqrt{98}, \lvert BC\rvert = \sqrt{49}, \lvert AC\rvert = \sqrt{147}$	M1A1	For S+ M1 for attempting one A1 for all 2 or 3 correct
	BC is shortest so must be side length	M1	Select shortest
	Volume $= 7^3 = 343$	A1 (5)	Requires all M marks
(b)	$\overrightarrow{PQ} \bullet \overrightarrow{PR} = 21 + 4 + 0 = 25$	M1	Attempt scalar product
	$\cos(QPR) = \dfrac{25}{\sqrt{50}\sqrt{25+\alpha^2}} = \dfrac{1}{2}$	M1	Use of cos 60 and scalar product formula to get an equation for α
	$\alpha = 5$ (Allow ± 5)	A1 (3)	
(c)	For $60°$ angle, $PQ = PR = \sqrt{50}$ must be a diagonal of a face	M1	Recognize PQ or PR is face diagonal. OK on fig.
	Therefore side must be 5 (since face diagonal is side$\times\sqrt{2}$)	A1	
	Diagonal is therefore $\underline{5\sqrt{3}}$	A1(3)	
		[11]	

Question 5 (AEA 2011 Q6)

The line L has equation

$$\mathbf{r} = \begin{pmatrix} 13 \\ -3 \\ -8 \end{pmatrix} + t\begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix}$$

The point P has position vector $\begin{pmatrix} -7 \\ 2 \\ 7 \end{pmatrix}$.

The point P' is the reflection of P in L.

(a) Find the position vector of P'.

(6)

(b) Show that the point A with position vector $\begin{pmatrix} -7 \\ 9 \\ 8 \end{pmatrix}$ lies on L.

(1)

(c) Show that angle $PAP' = 120°$.

(3)

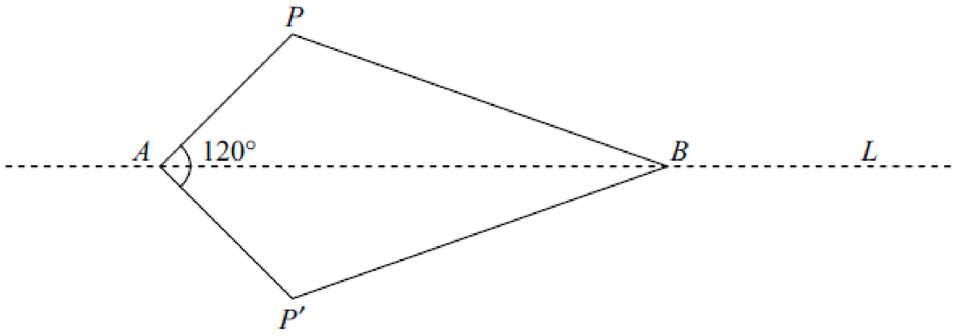


Figure 3

The point B lies on L and $APBP'$ forms a kite as shown in Figure 3.

The area of the kite is $50\sqrt{3}$

(d) Find the position vector of the point B.

(5)

(e) Show that angle $BPA = 90°$.

(2)

The circle C passes through the points A, P, P' and B.

(f) Find the position vector of the centre of C.

(2)

(a)	$\overrightarrow{PR}=\begin{pmatrix}13-5t--7\\-3+3t-2\\-8+4t-7\end{pmatrix}=\begin{pmatrix}20-5t\\-5+3t\\-15+4t\end{pmatrix}$	M1 A1	Attempt vector *PR*
	$\overrightarrow{PR}\bullet\begin{pmatrix}-5\\3\\4\end{pmatrix}=0\Rightarrow-100+25t-15+9t-60+16t=0$	M1	Attempt suitable scalar product
	$50t=175\Rightarrow t=\frac{7}{2}$	A1	
	If *X* is midpoint of PP' then $\overrightarrow{OP'}=\overrightarrow{OP}+2\overrightarrow{PX}$	M1	Strategy using known vectors NB *X* is $\left(-\frac{9}{2},\frac{15}{2},6\right)$
	$\overrightarrow{OP'}=\begin{pmatrix}-7\\2\\7\end{pmatrix}+2\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\\-1\end{pmatrix}=\begin{pmatrix}-2\\13\\5\end{pmatrix}$	A1 (6)	
(b)	Let $t=4$ then can see *A* lies on *L*	B1 (1)	Showing $t=4$ works
(c)	$\overrightarrow{AP}=\begin{pmatrix}0\\-7\\-1\end{pmatrix},\ \overrightarrow{AP'}=\begin{pmatrix}5\\4\\-3\end{pmatrix}\Rightarrow\overrightarrow{AP}\bullet\overrightarrow{AP'}=\frac{0-28+3}{\sqrt{50}\sqrt{50}}=-0.5$	M1 M1	Attempt suitable vectors(±) Attempt suitable scalar product (±)
	So $PAP'=120^o$	A1cso (3)	No incorrect working seen
(d)	$\lvert PP'\rvert=\sqrt{5^2+11^2+(-2)^2}=\sqrt{150}\left[=5\sqrt{6}\right]$	B1	
	Area $=\frac{1}{2}AB\times PP'=50\sqrt{3}\Rightarrow AB=10\sqrt{2}$ or $2\sqrt{50}$ o.e.	M1A1	Attempt $\lvert PP'\rvert$ (oe) or use sin60 M1 for attempt at equation giving length of *AB*
	$\left\lvert\overrightarrow{AX}\right\rvert=\frac{1}{2}\sqrt{50}$ so $\overrightarrow{AB}=4\overrightarrow{AX}$ or when $t=2$ in equation of *L*	M1	Strategy for finding *B*
	$\overrightarrow{OB}=\begin{pmatrix}3\\3\\0\end{pmatrix}$ $\left[\text{ignore } t=6\rightarrow\begin{pmatrix}-17\\15\\16\end{pmatrix}\right]$	A1 (5)	
(e)	$\overrightarrow{AP}=\begin{pmatrix}0\\-7\\-1\end{pmatrix},\ \overrightarrow{PB}=\begin{pmatrix}10\\1\\-7\end{pmatrix}\Rightarrow\overrightarrow{AP}\bullet\overrightarrow{PB}=0$ so angle is 90^o (*)	M1 A1cso (2)	Full method to find angle
(f)	Since *APB* is right angle *AB* is a diameter	M1	Using angle in semicircle theorem (S+ for mentioning)
	So centre is at midpoint $\frac{1}{2}\left[\begin{pmatrix}-7\\9\\8\end{pmatrix}+\begin{pmatrix}3\\3\\0\end{pmatrix}\right]=\begin{pmatrix}-2\\6\\4\end{pmatrix}$ or (-2,6,4)	A1 (2) **(19)**	
ALT (c)(d)	Finding $\overrightarrow{AP}$ and $\overrightarrow{AP'}$ $\left\lvert\overrightarrow{AP}\right\rvert=\left\lvert\overrightarrow{AP'}\right\rvert=\sqrt{50}$	M1 M1 B1	May show *PAB* = 60 B1 for $\lvert PP'\rvert$ from (d)

Question 6 (AEA 2010 Q4)

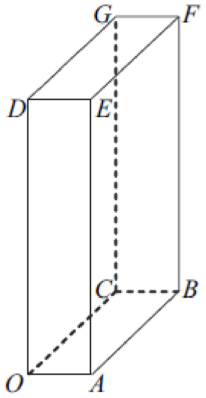


Figure 1

Figure 1 shows a cuboid *OABCDEFG*, where *O* is the origin, *A* has position vector $5\mathbf{i}$, *C* has position vector $10\mathbf{j}$ and *D* has position vector $20\mathbf{k}$.

(a) Find the cosine of angle *CAF*.

(4)

Given that the point *X* lies on *AC* and that *FX* is perpendicular to *AC*,

(b) find the position vector of point *X* and the distance *FX*.

(7)

The line l_1 passes through *O* and through the midpoint of the face *ABFE*. The line l_2 passes through *A* and through the midpoint of the edge *FG*.

(c) Show that l_1 and l_2 intersect and find the coordinates of the point of intersection.

(5)

(a)	$\overrightarrow{AC}=\begin{pmatrix}-5\\10\\0\end{pmatrix}, \overrightarrow{AF}=\begin{pmatrix}0\\10\\20\end{pmatrix}; \quad \lvert\overrightarrow{AC}\rvert=\sqrt{125}, \lvert\overrightarrow{AF}\rvert=\sqrt{500}$	B1 B1	Vectors *AC* or *AF*. Condone ± correct mods
	$\overrightarrow{AC}\bullet\overrightarrow{AF}=100 \Rightarrow \cos\angle CAF=\frac{100}{\sqrt{125}\sqrt{500}}, =\underline{\frac{2}{5}}$ or 0.4	M1 A1 (4)	Complete method for ± cos(*CAF*)
(b)	$\overrightarrow{OX}=\begin{pmatrix}5\\0\\0\end{pmatrix}+t\begin{pmatrix}-5\\10\\0\end{pmatrix}=\begin{pmatrix}5-5t\\10t\\0\end{pmatrix}$ $\underline{\text{or}}$ $\begin{pmatrix}a\\10-2a\\0\end{pmatrix}; \quad \overrightarrow{FX}=\begin{pmatrix}-5t\\10t-10\\-20\end{pmatrix}$	M1; M1	Attempt equation for *AC* or variable *OX* Attempt *FX*. Must be in terms of one unknown
	$\overrightarrow{FX}\bullet\overrightarrow{AC}=0 \Rightarrow 25t+100t-100+0=0, \quad [\,t=0.8\,]$	M1	Correct use of . to get linear eqn in *t*
	$\overrightarrow{OX}=\begin{pmatrix}1\\8\\0\end{pmatrix}; \quad \overrightarrow{FX}=\begin{pmatrix}-4\\-2\\-20\end{pmatrix}$ and $\lvert\overrightarrow{FX}\rvert=\sqrt{420}$	A1 A1 M1 A1 (7)	$t=0.8$ o.e. Correct vector *OX* Attempt ± *FX* $\sqrt{420}$ o.e.
	$[\lvert\overrightarrow{FX}\rvert=\sqrt{420}$ earns $\underline{\text{M1}}$ $\underline{\text{M1}}$ $\underline{\text{A1}}$; $\overrightarrow{OX}$ earns M1M1A1A1]		
(c)	$l_1:(\mathbf{r}=)\lambda\begin{pmatrix}5\\5\\10\end{pmatrix}$ and $l_2:(\mathbf{r}=)\begin{pmatrix}5\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2.5\\10\\20\end{pmatrix}$	B1 B1	B1 for each vector equation
	Solving: $5\lambda=5-2.5\mu$ and $5\lambda=10\mu$ (o.e.)	M1	Clear attempt to solve leading to $\lambda=$ or $\mu=$
	$\lambda=0.8, \quad \mu=0.4$	A1	Either Accept position vector
	Intersection at the point $\underline{\mathbf{(4, 4, 8)}}$	A1 (5) **[16]**	(S+ for clear attempt to check intersection)

Question 7 (AEA 2009 Q7)

Relative to a fixed origin O the points A, B and C have position vectors

$\mathbf{a} = -\mathbf{i} + \frac{4}{3}\mathbf{j} + 7\mathbf{k}$, $\mathbf{b} = 4\mathbf{i} + \frac{4}{3}\mathbf{j} + 2\mathbf{k}$ and $\mathbf{c} = 6\mathbf{i} + \frac{16}{3}\mathbf{j} + 2\mathbf{k}$ respectively.

(a) Find the cosine of angle ABC.

(3)

The quadrilateral $ABCD$ is a kite K.

(b) Find the area of K.

(3)

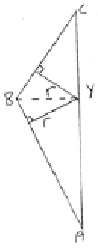
A circle is drawn inside K so that it touches each of the 4 sides of K.

(c) Find the radius of the circle, giving your answer in the form $p\sqrt{(q)} - q\sqrt{(p)}$, where p and q are positive integers.

(5)

(d) Find the position vector of the point D.

(7)

(a)	$\overrightarrow{BA} = \begin{pmatrix} -5 \\ 0 \\ 5 \end{pmatrix}$ $\quad \overrightarrow{BC} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ Attempt both	M1	Allow ±
	$\overrightarrow{BA} . \overrightarrow{BC} = -10 = 5\sqrt{2} \times 2\sqrt{5}\cos(A\hat{B}C)$ Use of .	M1	Use of . to form equation for cos $A\hat{B}C$
	$\therefore \quad \cos A\hat{B}C = -\frac{1}{\sqrt{10}}$ o.e.	A1 cso (3)	
(b)	Area of K = 2 Area of Δ ABC $= 2\times\frac{1}{2}\times 5\sqrt{2}\times 2\sqrt{5}\sin(A\hat{B}C)$	M1	Use of $\frac{1}{2}ab\sin C \times 2$
	$\sin(A\hat{B}C) = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$	M1	Attempt $\sin A\hat{B}C$ √ their (a)
	$\therefore \quad$ Area $= 5\sqrt{2}\times 2\sqrt{5}\times\frac{3}{\sqrt{10}} = \underline{30}$	A1 (3)	
(c)	 Identify $r \perp^r$ to BC and $r \perp^r$ to AB	B1	
	Area $= 2\times$ [Area of BYC + Area of BYA]	M1	Method → equation in r
	$30 = 2\times\left[\frac{1}{2}.2\sqrt{5}r + \frac{1}{2}.5\sqrt{2}r\right]$	A1	Correct equation in r
	$r = \frac{30}{2\sqrt{5}+5\sqrt{2}} = 30\frac{(5\sqrt{2}-2\sqrt{5})}{50-20}$	M1	Attempt r = with rational denom.
	$r = \underline{5\sqrt{2}-2\sqrt{5}}$	A1 (5)	

Question 8 (AEA 2008 Q7)

Relative to a fixed origin O, the position vectors of the points A, B and C are

$$\overrightarrow{OA} = -3\mathbf{i} + \mathbf{j} - 9\mathbf{k}, \quad \overrightarrow{OB} = \mathbf{i} - \mathbf{k}, \quad \overrightarrow{OC} = 5\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \text{ respectively.}$$

(a) Find the cosine of angle ABC.
(4)

The line L is the angle bisector of angle ABC.

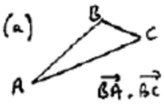
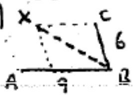
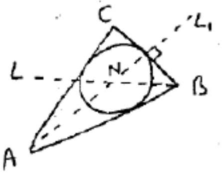
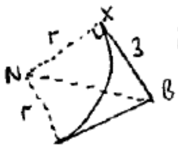
(b) Show that an equation of L is $\mathbf{r} = \mathbf{i} - \mathbf{k} + t(\mathbf{i} + 2\mathbf{j} - 7\mathbf{k})$.
(4)

(c) Show that $\lvert\overrightarrow{AB}\rvert = \lvert\overrightarrow{AC}\rvert$.
(2)

The circle S lies inside triangle ABC and each side of the triangle is a tangent to S.

(d) Find the position vector of the centre of S.
(7)

(e) Find the radius of S.
(5)

(a)  $\overrightarrow{BA} = \begin{pmatrix}-4\\1\\-8\end{pmatrix}$ $\quad \overrightarrow{BC} = \begin{pmatrix}4\\2\\-4\end{pmatrix}$ $\overrightarrow{BA}.\overrightarrow{BC} = -16+2+32 = 18$ $\lvert\overrightarrow{BA}\rvert = \sqrt{4^2+1^2+8^2} = 9$, $\lvert\overrightarrow{BC}\rvert = \sqrt{4^2+2^2+4^2} = 6$ $\therefore \quad \cos\theta = \frac{18}{9\times6} = \frac{1}{3}$	Attempt $\overrightarrow{BA}$ and $\overrightarrow{BC}$ Attempt $\overrightarrow{BA}.\overrightarrow{BC}$ Attempt $\lvert\overrightarrow{BA}\rvert$ or $\lvert\overrightarrow{BC}\rvert$	M1 M1 M1 A1 (4)
(b)  Using Rhombus idea, $\overrightarrow{BX} = \overrightarrow{BC} + \frac{2}{3}\overrightarrow{BA}$ o.e. $= \lambda\begin{pmatrix}4\\8\\-28\end{pmatrix}$ or $\mu\begin{pmatrix}1\\2\\-7\end{pmatrix}$ Through B $\therefore$ $\underline{r} = \begin{pmatrix}1\\0\\-1\end{pmatrix} + t\begin{pmatrix}1\\2\\-7\end{pmatrix}$ ⊛	eg $3\overrightarrow{BC} + 2\overrightarrow{BA}$ Any correct ratio	M1, A1 A1 A1 cso. (4)
(c) $\overrightarrow{AC} = \begin{pmatrix}8\\1\\4\end{pmatrix}$ $\quad \lvert\overrightarrow{AC}\rvert = \sqrt{8^2+1^2+4^2} = 9 = \lvert\overrightarrow{BA}\rvert$	Attempt $\overrightarrow{AC}$ and $\lvert\overrightarrow{AC}\rvert$ (must say $= \lvert\overrightarrow{BA}\rvert$ for A1)	M1 A1 cso (
		
(d) $\because$ ABC is isos L_1 has direction $\frac{1}{2}(\overrightarrow{AB}+\overrightarrow{AC}) = \begin{pmatrix}6\\0\\6\end{pmatrix}$ $\therefore$ L_1 has equation $(\underline{r} =) \begin{pmatrix}-3\\1\\-9\end{pmatrix} + u\begin{pmatrix}1\\0\\1\end{pmatrix}$ Centre of S is intersection of L_1 and L Solving: $\left.\begin{array}{l}1+t = -3+u\\2t = 1\end{array}\right\} \Rightarrow t = \frac{1}{2}, \quad u = 9/2$ $[\ -1-7t = -9+u \quad$ Check: LHS $= -\frac{9}{2}$, RHS $= -\frac{9}{2}\ ]$ $\therefore$ Centre has position vector $\overrightarrow{ON} = \begin{pmatrix}3/2\\1\\-9/2\end{pmatrix}$	 Find equation of L_1 Strategy Attempt to solve $t =$ or $u =$ (-1eoo)	B1 M1 M1 M1 A1 A2/1/0 (7)
(e)  Let X = mid-point of BC $\therefore$ BX = 3 ($\because$ isos) $\overrightarrow{BN} = \begin{pmatrix}1/2\\1\\-7/2\end{pmatrix}$ $\quad \therefore \overrightarrow{BN} = \frac{1}{2}\sqrt{54}$ $r^2 = BN^2 - 3^2 \quad \therefore r^2 =, \frac{54}{4} - 9. \quad \therefore r = \frac{\sqrt{18}}{2}$ or $\frac{3}{2}\sqrt{2}$	BX = 3 Attempt $\overrightarrow{BN}$ & $\lvert\overrightarrow{BN}\rvert$ Full method for r	B1 M1 A1 M1 A1 (5) (22)

Question 9 (AEA 2007 Q7)

The points O, P and Q lie on a circle C with diameter OQ. The position vectors of P and Q, relative to O, are $\mathbf{p}$ and $\mathbf{q}$ respectively.

(a) Prove that $\mathbf{p}.\mathbf{q} = |\mathbf{p}|^2$. **(3)**

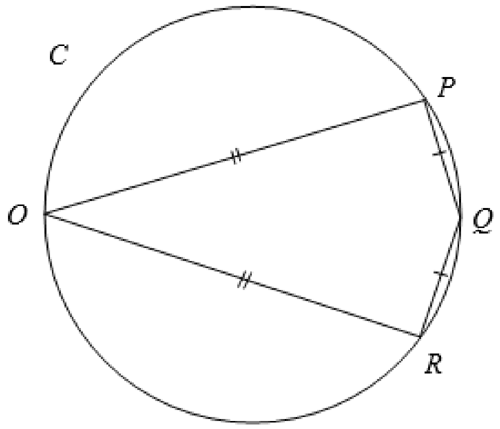


Figure 3

The point R also lies on C and $OPQR$ is a kite K as shown in Figure 3. The point S has position vector, relative to O, of $\lambda\mathbf{q}$, where λ is a constant. Given that $\mathbf{p} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{q} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and that OQ is perpendicular to PS, find

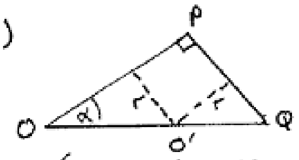
(*b*) the value of λ, **(2)**
(*c*) the position vector of R, **(3)**
(*d*) the area of K. **(4)**

Another circle C_1 is drawn inside K so that the 4 sides of the kite are each tangents to C_1.

(*e*) Find the radius of C_1 giving your answer in the form $(\sqrt{2} - 1)\sqrt{n}$, where n is an integer. **(5)**

A second kite K_1 is similar to K and is drawn inside C_1.

(*f*) Find that area of K_1. **(3)**

(a) $\overrightarrow{OQ}$ is diameter ∴ $O\hat{P}Q = 90°$ (∠ in semicircle)	Reason for $O\hat{P}Q = 90°$	B1	
∴ $\mathbf{p}\cdot(\mathbf{q}-\mathbf{p}) = 0$	Use of $\mathbf{a}.\mathbf{b} = 0$	M1	
⇒ $\mathbf{p}\cdot\mathbf{q} = \mathbf{p}\cdot\mathbf{p} = \lvert\mathbf{p}\rvert^2$ ⊛		A1	(3)
(b) $\overrightarrow{PS} \perp^r \overrightarrow{OQ}$ ⇒ $\mathbf{q}\cdot(\lambda\mathbf{q}-\mathbf{p}) = 0$	Full method → eqn in λ	M1	
∴ $\lambda \times 9 = \mathbf{p}\cdot\mathbf{q} = \lvert\mathbf{p}\rvert^2 = 6$ $\quad \underline{\lambda = 2/3}$		A1	(2)
(c) $\overrightarrow{OR} = \overrightarrow{OP} + 2\overrightarrow{PS}$	A valid vector route for $\overrightarrow{OR}$	M1	
$= \mathbf{p} + \frac{4}{3}\mathbf{q} - 2\mathbf{p} = \frac{4}{3}\mathbf{q} - \mathbf{p}$	Correct expression	A1	
$\overrightarrow{OR} = \begin{pmatrix} 5/3 \\ -2/3 \\ -5/3 \end{pmatrix}$		A1	(3)
(d) Area of K = 2 × ΔOPQ. ΔOPQ = $\frac{1}{2}\lvert\mathbf{q}\rvert\lvert\overrightarrow{PS}\rvert$ or $\frac{1}{2}\lvert\mathbf{p}\rvert\lvert\overrightarrow{PQ}\rvert$	Formula for area	M1	
$\lvert\mathbf{p}\rvert = \sqrt{6}$ or $\lvert\mathbf{q}\rvert = 3$	$\lvert\mathbf{p}\rvert$ or $\lvert\mathbf{q}\rvert$	B1	(suitably paired)
and $\lvert\overrightarrow{PQ}\rvert = \left\lvert\begin{matrix} 1 \\ -1 \\ -1 \end{matrix}\right\rvert = \sqrt{3}$ or $\lvert\overrightarrow{PS}\rvert = \left\lvert\begin{matrix} 1/3 \\ -4/3 \\ -4/3 \end{matrix}\right\rvert = \frac{\sqrt{18}}{3}$	$\lvert\overrightarrow{PQ}\rvert$ or $\lvert\overrightarrow{PS}\rvert$	B1	
∴ Area of K = $3 \times \frac{\sqrt{18}}{3}$ or $\sqrt{6}\times\sqrt{3}$ $= \underline{\sqrt{18} \text{ or } 3\sqrt{2}}$		A1	(4)
(e)  Identify and attempt to use square		M1	
$\tan\alpha = \frac{\lvert\overrightarrow{PQ}\rvert}{\lvert\overrightarrow{OP}\rvert} = \frac{r}{\lvert\overrightarrow{OP}\rvert - r}$			
$\left(\tan\alpha = \right) \frac{\sqrt{3}}{\sqrt{6}} = \frac{r}{\sqrt{6} - r}$	Forming an equation in r (attempt at)	M1	
	correct	A1	
⇒ $\sqrt{18} - \sqrt{3}r = \sqrt{6}r$ ∴ $r = \frac{\sqrt{18}}{\sqrt{6}+\sqrt{3}}$	Obtain expression r =	M1	
$= \underline{\sqrt{6}(\sqrt{2}-1)}$	$\Lambda = 6$	A1	(5)
(f) Radius of C is $\frac{1}{2}\overrightarrow{OQ} = 3/2$		B1	
Using ratio of area = (ratio of radii)².			
Area of $K_1 = \left[\frac{(e)}{3/2}\right]^2 \times (d) = \left(\left[\frac{2\sqrt{6}(\sqrt{2}-1)}{3}\right]^2 \times 3\sqrt{2}\right)$	Full method for area	M1	
$= \underline{8\sqrt{2}(\sqrt{2}-1)^2 \text{ or } 24\sqrt{2} - 32 \text{ or } 8(3\sqrt{2}-4)}$	or equivalent form with no more surds and simplified fractions	A1	(3)
			⑳

Question 10 (AEA 2006 Q5)

The lines L_1 and L_2 have vector equations

L_1: $\quad \mathbf{r} = -2\mathbf{i} + 11.5\mathbf{j} + \lambda(3\mathbf{i} - 4\mathbf{j} - \mathbf{k})$,

L_2: $\quad \mathbf{r} = 11.5\mathbf{i} + 3\mathbf{j} + 8.5\mathbf{k} + \mu(7\mathbf{i} + 8\mathbf{j} - 11\mathbf{k})$,

where λ and μ are parameters.

(*a*) Show that L_1 and L_2 do not intersect.

(5)

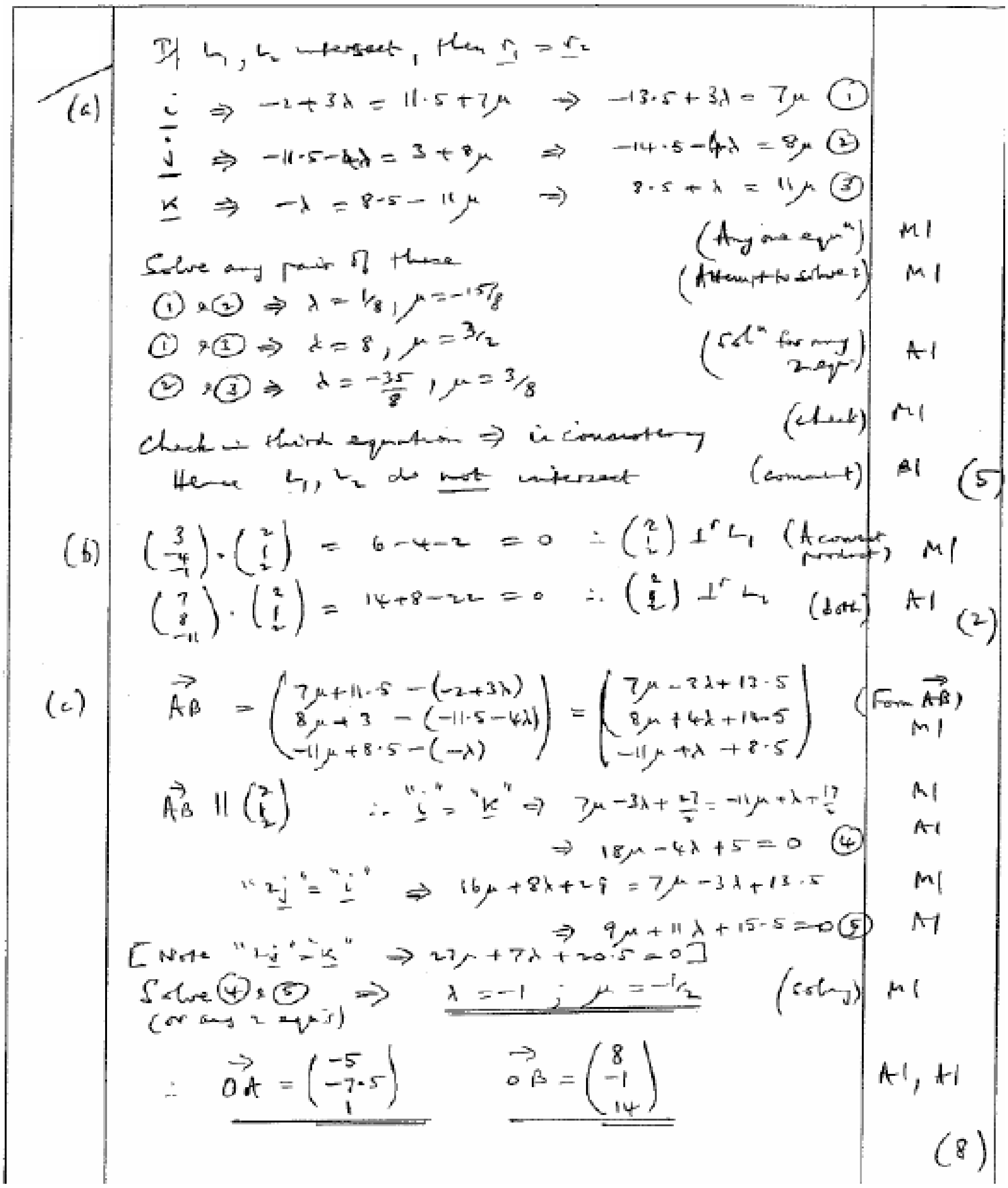
(*b*) Show that the vector $(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$ is perpendicular to both L_1 and L_2.

(2)

The point A lies on L_1, the point B lies on L_2 and AB is perpendicular to both L_1 and L_2.

(*c*) Find the position vector of the point A and the position vector of the point B.

(8)



(a) If L_1, L_2 intersect, then $\underline{r}_1 = \underline{r}_2$

$\underline{i} \Rightarrow -2+3\lambda = 11.5+7\mu \Rightarrow -13.5+3\lambda = 7\mu$ ①

$\underline{j} \Rightarrow -11.5-4\lambda = 3+8\mu \Rightarrow -14.5-4\lambda = 8\mu$ ②

$\underline{k} \Rightarrow -\lambda = 8.5-11\mu \Rightarrow 8.5+\lambda = 11\mu$ ③

(Any one eqⁿ) M1

Solve any pair of these (Attempt to solve 2) M1

① & ② $\Rightarrow \lambda = 1/8, \mu = -15/8$

① & ③ $\Rightarrow \lambda = 8, \mu = 3/2$ (Solⁿ for any 2 eqⁿs) A1

② & ③ $\Rightarrow \lambda = -\frac{35}{8}, \mu = 3/8$

Check in third equation $\Rightarrow$ inconsistency (check) M1

Hence L_1, L_2 do not intersect (comment) A1 (5)

(b) $\begin{pmatrix}3\\-4\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = 6-4-2 = 0 \therefore \begin{pmatrix}2\\1\\2\end{pmatrix} \perp^r L_1$ (A correct product) M1

$\begin{pmatrix}7\\8\\-11\end{pmatrix}\cdot\begin{pmatrix}2\\1\\2\end{pmatrix} = 14+8-22 = 0 \therefore \begin{pmatrix}2\\1\\2\end{pmatrix} \perp^r L_2$ (both) A1 (2)

(c) $\vec{AB} = \begin{pmatrix}7\mu+11.5-(-2+3\lambda)\\8\mu+3-(-11.5-4\lambda)\\-11\mu+8.5-(-\lambda)\end{pmatrix} = \begin{pmatrix}7\mu-3\lambda+13.5\\8\mu+4\lambda+14.5\\-11\mu+\lambda+8.5\end{pmatrix}$ (Form $\vec{AB}$) M1

$\vec{AB} \parallel \begin{pmatrix}2\\1\\2\end{pmatrix} \quad \therefore$ "$\underline{i}$" = "$\underline{k}$" $\Rightarrow 7\mu-3\lambda+\frac{27}{2} = -11\mu+\lambda+\frac{17}{2}$ M1

$\Rightarrow 18\mu-4\lambda+5=0$ ④ A1

"$2\underline{j}$" = "$\underline{i}$" $\Rightarrow 16\mu+8\lambda+29 = 7\mu-3\lambda+13.5$ M1

$\Rightarrow 9\mu+11\lambda+15.5=0$ ⑤ A1

[Note "$2\underline{j}$" = "$\underline{k}$" $\Rightarrow 27\mu+7\lambda+20.5=0$]

Solve ④ & ⑤ (or any 2 eqⁿs) $\Rightarrow \underline{\underline{\lambda=-1\ ;\ \mu=-1/2}}$ (solving) M1

$\therefore \underline{\underline{\vec{OA} = \begin{pmatrix}-5\\-7.5\\1\end{pmatrix}}} \qquad \underline{\underline{\vec{OB} = \begin{pmatrix}8\\-1\\14\end{pmatrix}}}$ A1, A1

(8)

Question 11 (AEA 2005 Q5)

The point A has position vector $7\mathbf{i} + 2\mathbf{j} - 7\mathbf{k}$ and the point B has position vector $12\mathbf{i} + 3\mathbf{j} - 15\mathbf{k}$.

(*a*) Find a vector for the line L_1 which passes through A and B.

(2)

The line L_2 has vector equation

$$\mathbf{r} = -4\mathbf{i} + 12\mathbf{k} + \mu(\mathbf{i} - 3\mathbf{k}).$$

(*b*) Show that L_2 passes through the origin O.

(1)

(*c*) Show that L_1 and L_2 intersect at a point C and find the position vector of C.

(3)

(*d*) Find the cosine of $\angle OCA$.

(3)

(*e*) Hence, or otherwise, find the shortest distance from O to L_1.

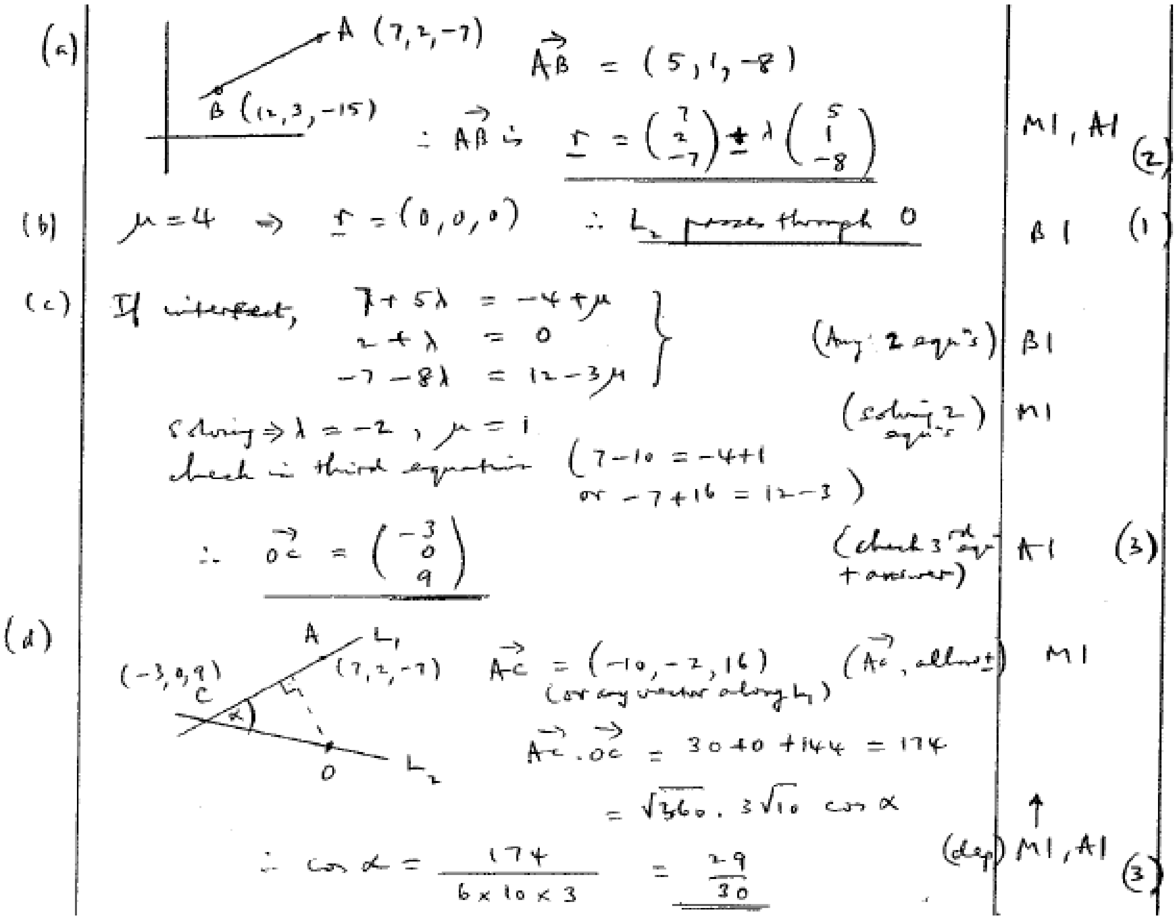
(3)

(*f*) Show that $|\overrightarrow{CO}| = |\overrightarrow{AB}|$.

(2)

(*g*) Find a vector equation for the line which bisects $\angle OCA$.

(5)



(a) A (7, 2, −7) B (12, 3, −15)

$\overrightarrow{AB} = (5, 1, -8)$

$\therefore \overrightarrow{AB}$ is $\underline{r} = \begin{pmatrix}7\\2\\-7\end{pmatrix} \pm \lambda\begin{pmatrix}5\\1\\-8\end{pmatrix}$ M1, A1 (2)

(b) $\mu = 4 \Rightarrow \underline{r} = (0, 0, 0)$ $\therefore$ L_2 passes through O B1 (1)

(c) If intersect, $7 + 5\lambda = -4 + \mu$, $2 + \lambda = 0$, $-7 - 8\lambda = 12 - 3\mu$ (Any 2 eqn's) B1

solving $\Rightarrow \lambda = -2$, $\mu = 1$ (solving 2 eqn's) M1

check in third equation ($7 - 10 = -4 + 1$ or $-7 + 16 = 12 - 3$)

$\therefore \overrightarrow{OC} = \begin{pmatrix}-3\\0\\9\end{pmatrix}$ (check 3rd eqn + answer) A1 (3)

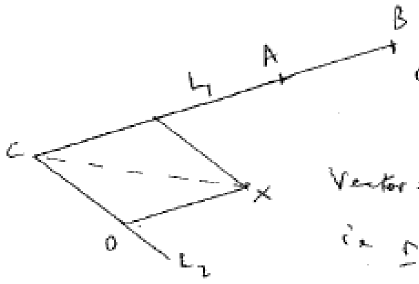
(d) A L₁ (−3, 0, 9) C (7, 2, −7) α O L₂

$\overrightarrow{AC} = (-10, -2, 16)$ (or any vector along L_1) ($\overrightarrow{AC}$, allow ±) M1

$\overrightarrow{AC} \cdot \overrightarrow{OC} = 30 + 0 + 144 = 174$

$= \sqrt{360} \cdot 3\sqrt{10} \cos\alpha$

$\therefore \cos\alpha = \dfrac{174}{6 \times 10 \times 3} = \dfrac{29}{30}$ (dep) M1, A1 (3)

(e)	Shortest distance $= \lvert OC \rvert \sin\alpha = 3\sqrt{10}\cdot\sqrt{1-\left(\frac{29}{30}\right)^2}$ $= 3\sqrt{10}\cdot\frac{\sqrt{59}}{3\sqrt{10}\sqrt{10}} = \underline{\underline{\sqrt{\frac{59}{10}}}}$ o.e.	(complete method for M2) M2, A1 (3)
(f)	$\lvert \overrightarrow{CO} \rvert = 3\sqrt{10}$; $\lvert \overrightarrow{AB} \rvert = \sqrt{25+1+64} = 3\sqrt{10}$ $\therefore \underline{\underline{\lvert \overrightarrow{CO} \rvert = \lvert \overrightarrow{AB} \rvert}}$	M1 (both lengths A1 (2)
(g)	 $\overrightarrow{CX} = \overrightarrow{CO} + \overrightarrow{OX} \left[= \overrightarrow{CO} + \overrightarrow{AB}\right]$ $= \begin{pmatrix} 3 \\ 0 \\ -9 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -17 \end{pmatrix}$ (dep) Vector eqⁿ of bisector $\underline{r} = \overrightarrow{OC} + t\,\overrightarrow{CX}$ i.e. $\underline{\underline{\underline{r} = \begin{pmatrix} -3 \\ 0 \\ 9 \end{pmatrix} \pm t \begin{pmatrix} 8 \\ 1 \\ -17 \end{pmatrix}}}$	M1 ↑ M1, A1 ↑ M1 (dep) A1 (5)